TOPICS IN PROBABILITY. PART II: UNIVERSALITY

Exercise sheet 8: Universality and strictly stable distributions

Exercise 1 (Characteristic functions of strictly stable laws)**.**

Let $\phi_{\alpha,a}(t) = \exp(-a|t|^\alpha)$*. Show that* $\phi_{\alpha,a}$ *is a characteristic function of a random variable iff* $a > 0$*,* $\alpha \in (0, 2]$*. Show that the latter ones are exactly characteristic functions of symmetric strictly stable* α *distributions.*

Remark: Note that there are strictly stable distributions whose characteristic exponent is more complicated and involves imaginary terms, e.g. standard Lévy distribution or generalized Cauchy distribution (with non-zero location parameter).

Exercise 2 (Strictly stable distribution: equivalent definition)**.**

1) Random variable X *has a strictly stable distribution iff the following condition holds: if* X_1, X_2 *are independent variables, each with the same distribution as* X *and* $a, b > 0$*, then there exists a constant* $c > 0$ *such that* $aX_1 + bX_2$ *has the same distribution as* cX*.*

Hint: for "only if" direction, find the explicit norming parameters using previous exercise, understand for which choices of a, b *one could directly conclude and how to extend the result to all* $a, b > 0$*.*

2) Prove that if X *has a strictly stable distribution with norming constant* c_n*, then so is the law of* $-X$*. Conclude that if* Y *is independent of* X *with the same distribution, then the distribution of* $X - Y$ *is strictly stable with the same norming constant. Note that law of* $X - Y$ *is as well symmetric w.r.t. 0.*

Exercise 3 (Heavy tailed CLT)**.**

Consider a random variable X*, whose law is symmetric around zero and that satisfies* $\mathbb{P}[|X| > x] = x^{-\alpha}$ *for* $\alpha \in (0, 2)$ *for all* $x \geq 1$*. Now let* $X_1, X_2, \dots$ *be i.i.d. with the law of* X*. E.g. by using characteristic functions (or otherwise) prove that* $\frac{1}{n^{1/\alpha}} \sum_{i=1}^n X_i$ *converges to the symmetric* α *strictly stable law, i.e. the norming constant is* $c_n = n^{1/\alpha}$ *and the law is symmetric w.r.t. zero.*

Exercise 4 ("Open question")**.** *Find classes of functions* $f_n : \mathbb{R}^n \to \mathbb{R}$ *such that* $f_n \in C^\infty$ *and so that*

$$\sup_x \left| \frac{\partial^3}{\partial x_i^3} f_n(x) \right| = o(n^{-1}) \quad \forall 1 \leq i \leq n,$$

for all n *sufficiently large*