Modular forms and applications

Eta, theta,

and partitions

Partitions

A *partition* of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition.

The *partition function* $p(n)$ represents the number of possible partitions of a natural number n.

Example

$5 \; = \; 4+1 \; = \; 3+2 \; = \; 3+1+1 \; = \; 2+2+1 \; = \; 2+1+1+1 \; = \; 1+1+1+1+1$
$p(5) = 7$

Lemma

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty}(1 + x^k + x^{2k} + \dots) = \prod_{k=1}^{\infty}(1 - x^k)^{-1}.$$

Definition

For a positive integer n we denote by $p_e(n)$ the number of partitions of n into an even number of *different* parts, analogously, we denote by $p_o(n)$ the number of partitions of n into an odd number of *different* parts.

Lemma

We have

$$\prod_{k=1}^{\infty}(1 - x^k) = \sum_{s=0}^{\infty} \sum_{k_1<k_2<\cdots<k_s} (-1)^s\, x^{k_1+k_2+\ldots+k_s} = \sum_{n=0}^{\infty} c_n\, x^n$$

where $c_0 = 1$ and for $n \geq 1$ and $c_n = p_e(n) - p_o(n)$.

Let us do an experiment

We compute $\prod_{i=1}^{20}(1-x^i)+o(x^{20})$.
The answer is $1-x-x^2+x^5+x^7-x^{12}-x^{15}+O\left(x^{20}\right)$.

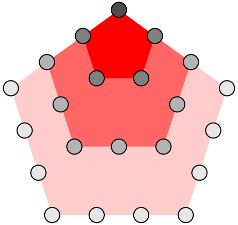
n		$p_o(n)$	$p_e(n)$
1	1	1	0
2	2	1	0
3	$3=2+1$	1	1
4	$4=3+1$	1	1
5	$5=4+1=3+2$	1	2
6	$6=5+1=4+2=3+2+1$	2	2
7	$7=6+1=5+2=4+3=4+2+1$	2	3

Pentagonal number theorem

Theorem(Euler)

$$\prod_{i=1}^{\infty}(1-x^i) = 1+\sum_{k=1}^{\infty}(-1)^k\left(x^{(3k^2-k)/2}+x^{(3k^2+k)/2}\right) = \sum_{k\in\mathbb{Z}}(-1)^k x^{(3k^2-k)/2}.$$

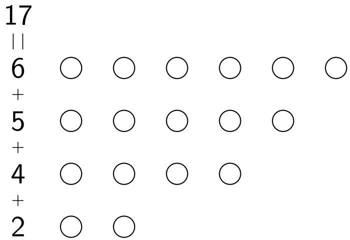
Pentagonal numbers



k	-3	-2	-1	0	1	2	3
$\frac{3k^2-k}{2}$	15	7	2	0	1	5	12

Proof

Consider a diagram of a partition of n into different parts.



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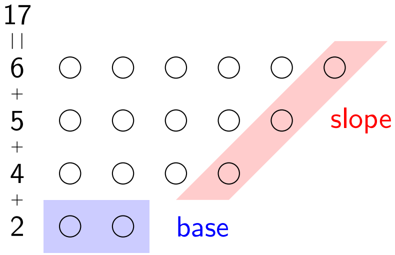

We define the following two operations on the diagrams and denote them α and β.

α:

If $|b| \leq |s|$ and if b and s have no common point, or if $|b| \leq |s| - 1$
Then we move b up and turn its points into new slope.

β:

If $|b| > |s|$ and if b and s have no common point, or if $|b| \geq |s| + 2$
Then we move the slope down and turn its points into the new base.

If b and s have a common point and $|b| = |s|$ or $|b| = |s| + 1$ then neither α no β can be applied.

α

β

To each diagram D we can apply at most one of the operations α or β.

If α can be applied to D then β can be applied to $\alpha(D)$.

If β can be applied to D then α can be applied to $\beta(D)$.

Thus we have a 1:1 correspondence

$$\left\{\begin{array}{c}\text{odd partitions}\\ \text{into distinct parts}\\ \text{s. t. } \alpha \text{ or } \beta \text{ can be applied}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{even partitions}\\ \text{into distinct parts}\\ \text{s. t. } \alpha \text{ or } \beta \text{ can be applied}\end{array}\right\}$$

When can neither α nor β be applied?

- b and s have a common point and $|b| = |s| = k$

$$n = k + (k+1) + \ldots + (2k-1) = \frac{3k^2 - k}{2}.$$

- b and s have a common point and $|b| - 1 = |s| = k$

$$n = (k+1) + \ldots + 2k = \frac{3k^2 + k}{2}.$$

Therefore:

$$p_e(n) - p_o(n) = c_n = \begin{cases} (-1)^k, & \text{if } n = \frac{3k^2 \pm k}{2} \\ 0, & \text{otherwise.} \end{cases}$$

This finishes the proof.

Jacobi triple product

Theorem

For all $z \neq 0$ and q with $|q| < 1$:

$$\prod_{k=1}^{\infty} (1 - q^{2k})\,(1 - q^{2k-1}\,z^2)\,(1 - q^{2k-1}\,z^{-2}) = \sum_{k=-\infty}^{\infty} (-1)^k\, z^{2k}\, q^{k^2}.$$

Proof

Set $\phi_n(z,q) := \prod_{k=1}^{n}(1-q^{2k-1}\,z^2)\,(1-q^{2k-1}\,z^{-2})$. We have

$$\phi_n(qz,q) \;=\; \phi_n(z,q)\,\frac{(1-q^{2n+1}\,z^2)\,(1-q^{-1}z^{-2})}{(1-qz^2)\,(1-q^{2n-1}z^{-2})} \;=\; \phi_n(z,q)\,\frac{1-q^{2n+1}\,z^2}{-qz^2+q^{2n}}. \qquad (1)$$

We consider the following expansion

$$\phi_n(z,q) = \sum_{k=-n}^{n} A_{n,k}(q)\,z^{2k}, \qquad (2)$$

where $A_{n,k}(q)$ is a polynomial in q. We have a symmetry $A_{n,k}(q) = A_{n,-k}(q)$. We find

$$A_{n,n}(q) = q^{1+3+5+\ldots+(2n-1)} = q^{n^2}. \qquad (3)$$

We substitute (2) into (1) and obtain

$$A_{n,k}(q) = A_{n,k-1}(q)\,q^{2k-1}\,\frac{1-q^{2n-2k+2}}{1-q^{2n+2k}}. \qquad (4)$$

From (3) and (4) we find

$$A_{n,k} = \frac{q^{k^2}}{(1-q^2)\cdots(1-q^{2n})} \prod_{s=n-k+1}^{n} (1-q^{2s}) \prod_{s=n+k+1}^{2n} (1-q^{2s}). \tag{5}$$

From (5) we obtain

$$\prod_{k=1}^{n} (1-q^{2k})\,(1-q^{2k-1}\,z^2)\,(1-q^{2k-1}\,z^{-2}) = \sum_{k=-n}^{n} (-1)^k z^{2k}\,q^{k^2}\,B_{n,k}(q).$$

where

$$B_{n,k}(q) = \prod_{s=n-k+1}^{n} (1-q^{2s}) \prod_{s=n+k+1}^{2n} (1-q^{2s}).$$

We have $B_{n,k} = 1 + O(q^{n-k+1})$. We take the limit $n \to \infty$ and finish the proof.

Question:

Can we deduce pentagonal number theorem from the Jacobi triple product theorem?

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Yes

Exercise

Find the necessary substitution.

Dedekind eta function

Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i z}.$$

We have

$$\eta(z) = q^{\frac{1}{24}} \sum_{k \in \mathbb{Z}} (-1)^k \, q^{\frac{3k^2+k}{2}} = \sum_{k \in \mathbb{Z}} (-1)^k \, q^{\frac{3}{2}(k+\frac{1}{6})^2} = \sum_{n=1}^{\infty} \chi(n) \, e^{\frac{1}{12}\pi i n^2 z},$$

where $\chi(n)$ is the Dirichlet character modulo 12 with $\chi(\pm 1) = 1$, $\chi(\pm 5) = -1$.

Dedekind eta function is a modular form of weight $1/2$.

Transformation properties of η

$$\eta(\tau+1) = e^{\pi i/12}\,\eta(\tau)$$
$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

Exercise

Show that the *Ramanujan delta function* $\Delta := \eta^{24}$ belongs to $M_{12}(\Gamma_1)$.
Show that $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$.

Exercise

Using the fact that $E_2(z) = \frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)}$ show that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2\,E_2(z) - \pi i c(cz+d).$$

A *weakly-holomorphic modular form* of weight k and congruence subgroup Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$
2. for each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ has the Fourier expansion $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{n \gg -\infty}^{\infty} c_\alpha(n)\, e^{2\pi i \frac{n}{h} z}$ for some $h \in \mathbb{N}$.

We denote the space of weakly-holomorphic modular forms of weight k and group Γ by $M_k^!(\Gamma)$. The spaces $M_k^!(\Gamma)$ are infinite dimensional.

Hardy-Ramanujan asymptotic formula

An asymptotic expression for $p(n)$ is given by

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \text{ as } n \to \infty.$$

Convergent series expansion for partition function

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\, A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right]\right).$$