# MODULARITY OF THETA FUNCTIONS OF INTEGRAL WEIGHT

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Abstract. We give a self-contained proof of the fact that theta functions attached to even lattices in even dimensional Euclidean spaces, weighted by harmonic polynomials, are modular forms of level equal to the level of the lattice, with respect to a quadratic (Dirichlet-) character. This is a classical result due to Hecke and Schoeneberg. Our exposition partially follows Schoeneberg [4] and Köcher-Krieg [2]. We also discuss an alternative approach using a Weil-type representation of $\mathrm{SL}_2(\mathbb{Z})$ on the discriminant groups of such lattices, describing the modularity of vector-valued theta functions, whose components are shifted theta functions.

## 1. Motivation

It is well-known that theta series attached to positive definite integral quadratic forms exhibit modular behavior. They give many examples of modular forms. In this context, a relatively rich and relatively simple class of quadratic forms are the positive definite integral quadratic forms in an *even* number of variables with *even* integral Gram matrices. By expressing these theta series as linear combination of Eisenstein series, one obtains formulas for the representation numbers of these quadratic forms.

The goal of these notes is to write up a complete, self-contained and (relatively) direct proof of the fact that the theta series attached to those types of quadratic forms (weighted by a harmonic polynomial) define modular forms. More precisely, we will show that that for all even integers $n \geq 2$, all even lattices $L \subseteq \mathbb{R}^n$ of level $N \geq 2$ and all harmonic polynomials $P$ on $\mathbb{R}^n$ of even degree $\delta \geq 0$, the theta series $\Theta_{L,P}(z) = \sum_{v \in L} P(v) e^{\pi i |v|^2 z}$ defines a modular form in $M_k(\Gamma_0(N), \chi) \subseteq M_k(\Gamma_1(N))$, where $k = n/2 + \delta$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ a quadratic Dirichelt character depending only on $k$, $N$ and the discriminant $\operatorname{disc} L$. We give precise statements in Theorems 1, 2 and 3, which we prove in §3, after fixing notation and gather some auxiliary results in §2.

Our exposition is based on the books [4] and [2], with some variation and simplifications here and there. The main result (Theorem 3) is often attribued to Erich Hecke and Bruno Schoeneberg, see [5] and the references therein.

## 2. Notations and preliminary facts

We collect a few basic definitions and facts that will be used in the sequel. Readers may also go directly to §3 and refer back to this section as necessary.

2.1. **Lattices.** We denote the standard Euclidean inner product between vectors $x, y \in \mathbb{R}^n$ by $\langle x, y\rangle$ and we write $|x| = \sqrt{\langle x, x\rangle}$ for the associated norm. Let $L \subseteq \mathbb{R}^n$ be a lattice with dual lattice

$$L^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \in \mathbb{Z}\, \forall x \in L\} \subseteq \mathbb{R}^n.$$

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Recall that $L$ is said to be *integral*, if $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in L$ and *even*, if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. The latter property implies the former. If $L$ is even, we define the *level* of $L$ to be the smallest positive integer $N \geq 1$ such that $N^{1/2}L^*$ is again an even lattice. Given a $\mathbb{Z}$-basis $e_1, \dots, e_n$ of $L$, the associated Gram matrix is the symmetric, positive definite matrix $Q = (\langle e_i, e_j\rangle)_{1\leq i,j\leq n}$. It depends upon the basis of $L$ and changing the basis changes the Gram matrix $Q$ to $B^tQB$, for some $B \in \mathrm{GL}_n(\mathbb{Z})$. It follows that the *discriminant* $\operatorname{disc} L := \det Q > 0$ is well-defined (in the sense that it depends only upon $L$ and not upon the basis of $L$). If $L$ is integral, then $\operatorname{disc} L$ is a positive integer. Also, if $L$ is integral, then any Gram matrix of $L$ has integral entries and if $L$ is even, then the diagonal entries are in addition even integers. The *covolume* of a lattice $L \subseteq \mathbb{R}^n$ is the absolute value of the determinant of any $\mathbb{Z}$-basis of $L$, assembled as rows (or columns) into a matrix. We denote it as $|L|$ or $\mathrm{covol}(L)$. It also equals measure of the quotient group $\mathbb{R}^n/L$ (taking the counting measure on $L$ and the standard Haar measure on $\mathbb{R}^n$ which assigns volume 1 to $[0,1]^n$) and we have $|L|^2 = \operatorname{disc} L$ . Indeed, if $B\mathbb{Z}^n = L$, for some $B \in \mathrm{GL}_n(\mathbb{R})$, then $B^tB$ is a Gram matrix of $L$.

2.2. **Fourier transforms, Gaussians, Poisson summation.** For any integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we define its Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx.$$

We will need the following well-known fact.

**Lemma 2.1.** *Let $P : \mathbb{R}^n \to \mathbb{C}$ be a polynomial which is homogeneous of degree $\delta \geq 0$ and annihilated by the Laplacian $\Delta$, i.e. such that $\Delta P = 0$. Then*

$$\int_{\mathbb{R}^n} P(x)e^{-\pi|x|^2}e^{-2\pi i\langle x,\xi\rangle}d\xi = (-i)^\delta P(\xi)e^{-\pi|\xi|^2}. \tag{2.1}$$

*Proof.* This is easy if $n = 1$, so assume $n \geq 2$. The space $\mathcal{H}_\delta(\mathbb{R}^n)$ consisting of all $P$ as in the statement is a finite dimensional, irreducible representation for the orthogonal group $\mathrm{O}(n)$ (acting by precomposition). From this, one deduces that it suffices to verify (2.1) for a single nonzero $P \in \mathcal{H}_\delta(\mathbb{R}^n)$. A computation gives the claim for $P(x) = (x_1 + ix_2)^\delta$. □

**Remark 2.1.** Lemma 2.1 says that $P(x)e^{-\pi|x|^2}$ is an eigenvector for the Fourier transform with eigenvalue $(-i)^\delta$. The eigenvalue depends on choices of normalization, in particular, on the minus sign in $-2\pi i\langle x, \xi\rangle$ in our normalization of the Fourier transform. If we change the sign in this normalization, the eigenvalue is instead $i^\delta$. Note that if $\delta$ is even, there is no difference between the two eigenvalues.

We shall need a version of Lemma 2.1, in which the Gaussian is "scaled" by a complex number in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. In the following (unless specified otherwise), for $z \in \mathbb{H}$ and $\alpha \in \mathbb{C}$, the complex number $(z/i)^\alpha$ is defined as $(z/i)^\alpha := \exp(\alpha \log(z/i))$, where we fix the holomorphic logarithm $z \mapsto \log(z/i)$ by requiring that it equals 0 at the point $i$.

**Corollary 2.1.** *If $P$ is as in Lemma 2.1 and $z \in \mathbb{H}$, then*

$$\int_{\mathbb{R}^n} P(x)e^{\pi iz|x|^2}e^{-2\pi i\langle x,\xi\rangle}d\xi = (-i)^\delta(z/i)^{-n/2-\delta}P(\xi)e^{-\pi|\xi|^2}. \tag{2.2}$$

*Proof.* Define $f_z(x) := P(x)e^{\pi iz|x|^2}$. We need to show that

$$\widehat{f_z}(\xi) = (-i)^\delta(z/i)^{-\delta-n/2}P(\xi)e^{\pi i(-1/z)|\xi|^2},$$

for all $\xi \in \mathbb{R}^n$ and $z \in \mathbb{H}$. For fixed $\xi \in \mathbb{R}^n$, both sides of this identity depend holomorphically upon $z \in \mathbb{H}$, so it suffices to prove that

$$\widehat{f_{it}}(\xi) = (-i)^\delta t^{-\delta-n/2} P(\xi) e^{\pi(-1/t)|\xi|^2},$$

for all $t > 0$ and all $\xi \in \mathbb{R}^n$. We have

$$\begin{aligned}
\widehat{f_{it}}(\xi) &= \int_{\mathbb{R}^n} P(x) e^{-\pi t|x|^2} e^{-2\pi i\langle x,\xi\rangle} dx \quad \text{(by definition)} \\
&= t^{-n/2} \int_{\mathbb{R}^n} P(t^{-1/2}t^{1/2}x) e^{-\pi|t^{1/2}x|^2} e^{-2\pi i\langle t^{1/2}x, t^{-1/2}\xi\rangle} t^{\delta n/2} dx \quad \Big(\text{writing } x = t^{-1/2}t^{1/2}x\Big) \\
&= t^{-n/2} \int_{\mathbb{R}^n} P(t^{-1/2}y) e^{-\pi|y|^2} e^{-2\pi i\langle y, t^{-1/2}\xi\rangle} dy \quad \Big(\text{changing variables } y = t^{1/2}x\Big) \\
&= t^{-n/2-\delta/2} \int_{\mathbb{R}^n} P(y) e^{-\pi|y|^2} e^{-2\pi i\langle y, t^{-1/2}\xi\rangle} dy \quad \text{(using that } P \text{ is homogeneous)} \\
&= t^{-n/2-\delta/2} i^{-\delta} P(t^{-1/2}\xi) e^{-\pi|t^{-1/2}\xi|^2} \quad \text{(using (2.1))} \\
&= i^{-\delta} t^{-n/2-\delta} P(\xi) e^{-\pi(1/t)|\xi|^2} \quad \text{(using that } P \text{ is homogeneous)},
\end{aligned}$$

as desired. □

**Corollary 2.2.** *Let $L \subseteq \mathbb{R}^n$ be a lattice and let $P : \mathbb{R}^n \to \mathbb{C}$ be as in Lemma 2.1. Then, for all $z \in \mathbb{H}$, we have*

$$\sum_{\lambda \in L} P(\lambda) e^{\pi i|\lambda|^2 z} = (-i)^\delta (z/i)^{-n/2-\delta} \frac{1}{|L|} \sum_{\lambda \in L^*} P(\lambda) e^{\pi i|\lambda|^2(-1/z)}.$$

*Proof.* This follows from Corollary 2.1 and the Poisson summation formula, which says that for any Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$, one has

$$\sum_{\lambda \in L} f(\lambda) = \frac{1}{|L|} \sum_{\lambda \in L^*} \widehat{f}(\lambda).$$

We apply it to $f(x) = P(x)e^{\pi iz|x|^2}$ and use Corollary 2.1. □

2.3. **The modular group.** Given a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we will use the notation $c_\gamma = c$, $d_\gamma = d$. For each $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we define a function

$$\mu(\gamma) : \mathbb{H} \to \mathbb{C}^\times, \qquad \mu(\gamma)(z) := \mu(\gamma, z) := c_\gamma z + d_\gamma.$$

Then we have the cocycle property $\mu(\gamma_1\gamma_2, z) = \mu(\gamma_1, \gamma_2 z)\mu(\gamma_2, z)$ for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$ and all $z \in \mathbb{H}$. It can also be written as $\mu(\gamma_1\gamma_2) = (\mu(\gamma_1) \circ \gamma_2) \cdot \mu(\gamma_2)$ (by slight abuse of notation). We use the following abbreviations for generators of the group $\mathrm{SL}_2(\mathbb{Z})$:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The defining relations for $\mathrm{SL}_2(\mathbb{Z})$ are $S^2 = (ST)^3$, $S^4 = I$, while those of $\mathrm{PSL}_2(\mathbb{Z})$ are[1] $S^2 = 1 = (ST)^3$. For any positive integer $N \geq 1$ we have the congruence subgroups

$$\Gamma_1(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : c_\gamma \equiv 0, a_\gamma \equiv d_\gamma \equiv 1 \pmod N\} \lhd \Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : c_\gamma \equiv 0 \pmod N\}$$

[1] where $S$ and $T$ should now be replace with their images in $\mathrm{PSL}_2(\mathbb{Z})$.

2.4. **Gauss sums, quadratic residue symbols.** Here, we recall a few definitions and facts about Gauss sums and quadratic residue symbols.

For all $a, q \in \mathbb{Z}$, $q \neq 0$, we define the Gauss sum $G(a,q) = \sum_{x\in\mathbb{Z}/q\mathbb{Z}} e(ax^2/q)$, where $e(r) := e^{2\pi i r}$. We also write $e_q(r) := e(r/q)$ sometimes. For odd and positive $q \geq 1$ with prime factorization $q = \ell_1 \cdots \ell_r$ (no necessarily distinct primes $\ell_i$) and any $a \in \mathbb{Z}$, we define the Jacobi symbol

$$\left(\frac{a}{q}\right) = \prod_{i=1}^{r} \left(\frac{a}{\ell_i}\right),$$

where the factors in the product are the Legendre symbols. For all odd $q \geq 1$ and all $a$ coprime to $q$, one has

$$G(a,q) = \left(\frac{a}{q}\right) G(1,q) = \left(\frac{a}{q}\right) \sqrt{q}\varepsilon_q,$$

where

$$\varepsilon_q := \begin{cases} 1 & q \equiv 1 \pmod 4, \\ i & q \equiv 3 \pmod 4. \end{cases} \tag{2.3}$$

We recall that $q \mapsto \varepsilon_q^2$ is the non-trivial Dirichlet character mod 4 and that $\varepsilon_p^2 = (-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right)$. For all odd, positive and coprime integers $p, q$ we we have the (generalized) quadratic reciprocity law

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right).$$

By factoring $p$ an $q$, it reduces to the case of distinct odd primes.

## 3. Theta functions and their modular behavior

We fix an integer $k \geq 1$, set $n := 2k$ and fix a positive definite, symmetric $n$-by-$n$ matrix $Q$ with integral entries and *even* diagonal entries. Given $v \in \mathbb{R}^n$, we write $Q[v] := v^tQv$. We fix some integer $N \geq 2$ such that $NQ^{-1}$ has integral entries with even diagonal entries.[2] Recall that $Q$ and $Q^{-1}$ admit unique, positive definite, symmetric square roots, which we denote by $Q^{1/2}$ and $Q^{-1/2}$ respectively; thus, we have $Q[v] = |Q^{1/2}v|^2$ for all $v \in \mathbb{R}^n$. We fix a harmonic polynomial $P : \mathbb{R}^n \to \mathbb{C}$, homogeneous of degree $\delta \geq 0$. To this data, we attach the *theta function*

$$\Theta(z) := \Theta_{Q,P}(z) := \sum_{v\in\mathbb{Z}^n} P(Q^{1/2}v)e^{\pi i z Q[v]} = \sum_{\lambda\in Q^{1/2}\mathbb{Z}^n} P(\lambda)e^{\pi i|\lambda|^2 z}, \qquad z \in \mathbb{H}.$$

Our goal is to study how $\Theta$ transforms under $\mathrm{SL}_2(\mathbb{Z})$ and to show that it is modular of weight $k+\delta$ on $\Gamma_1(N)$ and modular on $\Gamma_0(N)$ with respect to a certain character mod $N$.

3.1. **First step: modularity with respect to a character.** Our first goal is to prove the following theorem. We will refine it in the next section.

[2]We restrict to $N \geq 2$ so that we can use the fact that $d_\gamma \neq 0$ for all $\gamma \in \Gamma_0(N)$. If $Q^{-1}$ is also integral with even diagonal entries, then it is a well-known fact that $k \equiv 0 \pmod 4$ and that the theta functions attached to $Q$ are in $M_k(\mathrm{SL}_2(\mathbb{Z}))$ or $S_{k+\delta}(\mathrm{SL}_2(\mathbb{Z}))$ if they are weighted by a harmonic polynomial of degree $\delta > 0$, see Theorem 6.

**Theorem 1.** *In the above notation, the following holds. Suppose that $\Theta = \Theta_{Q,P}$ does not vanish identically (in particular, the degree $\delta$ of $P$ is even[3]). Then there exists a unique character $\psi : \Gamma_0(N) \to \mathbb{C}^\times$, depending only on $Q$, $k$ and $N$, such that*

$$\Theta\left(\frac{az+b}{cz+d}\right) = \psi(\gamma)(cz+d)^{k+\delta}\Theta(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), z \in \mathbb{H}. \tag{3.1}$$

*A formula for $\psi(\gamma)$ as a (generalized) Gauss sum is given in* (3.6).

It is remarkable that the transformation properties of $\Theta_{Q,P}$ depend on $P$ only through the degree $\delta$ of $P$. The proof of Theorem 1 will occupy the remainder of this section. Note that we can write (3.1) as

$$\Theta(\gamma z) = \psi(\gamma)\mu(\gamma, z)^{k+\delta}\Theta(z). \tag{3.2}$$

The *uniqueness* of the character $\psi$ is clear, because if the above holds for $\psi = \psi_1, \psi_2$, then, upon taking differences, we get $0 = (\psi_1(\gamma) - \psi_2(\gamma))\mu(\gamma, z)^{k+\delta}\Theta(z)$ for all $\gamma \in \Gamma_0(N)$ and all $z \in \mathbb{H}$. Since $\Theta$ does not vanish identically by assumption, this implies $\psi_1(\gamma) = \psi_2(\gamma)$.

Now we turn to the existence proof of $\psi$, which will culminate in the formula (3.6), after two-fold application of Poisson summation and a rearrangement of sums in between. Consider an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Note that we always have $d \neq 0$, since $N \geq 2$ by assumption. If $c = 0$, then $\gamma z = z + m$ for some $m \in \mathbb{Z}$ and hence $\Theta(\gamma z) = \Theta(z + m) = \Theta(z)$ since $Q$ is even.We also note that (3.2) implies that $\psi$ should have the property $\psi(-I) = (-1)^k$. We keep this in mind and now assume $c \neq 0$. A key ingredient in the proof is the following elementary identity

$$\gamma z = \frac{a}{c} - \frac{1}{c(cz+d)} = \frac{a}{c} - \frac{1}{c^2\tau} = \frac{a}{c} + \frac{1}{c^2}(-1/\tau), \quad \text{where} \quad \tau := z + d/c \in \mathbb{H}. \tag{3.3}$$

Using (3.3), we can write

$$\Theta(z) = \sum_{v \in \mathbb{Z}^n} P(Q^{1/2}v) e^{\pi i (a/c) Q[v]} e^{\pi i (-1/\tau) Q[v/c]}.$$

We will divide the $v$-sum into residue classes mod $c\mathbb{Z}^n$. We observe that for all $r, h \in \mathbb{Z}^n$, we have

$$Q[r + hc] = Q[r] + 2cr^t h + Q[c] \equiv Q[r] \pmod{2c}, \quad \text{and} \quad P(Q^{1/2}(r + hc)) = c^\delta P(Q^{1/2}(h + r/c)),$$

We temporarily fix a complete of coset representatives $R_c \subseteq \mathbb{Z}^n$ for $\mathbb{Z}^n / c\mathbb{Z}^n$. We write $v = ch + r$ with $h \in \mathbb{Z}^n$, and $r \in R_c$ to get

$$\Theta(\gamma z) = c^\delta \sum_{r \in R_c} e^{\pi i (a/c) Q[r]} \sum_{h \in \mathbb{Z}^n} P(Q^{1/2}(h + r/c)) e^{\pi i (-1/\tau) Q[h + r/c]}.$$

Applying Poisson summation to the inner sum, we get

$$\Theta(\gamma z) = c^\delta (-i)^\delta (\tau/i)^{k+\delta} (\det Q)^{-1/2} \sum_{r \in R_c} e^{\pi i (a/c) Q[r]} \sum_{v \in \mathbb{Z}^n} P(Q^{-1/2}v) e^{\pi i \tau Q^{-1}[v]} e^{2\pi i v^t (r/c)}.$$

[3] Note that, by changing $v$ to $-v$ in the sum defining $\Theta$, we obtain $\Theta(z) = (-1)^\delta \Theta(z)$.

Recalling that $\tau = z + d/c$, we rewrite this as

$$\Theta(\gamma z) = \Big(c^{\delta}(-i)^{\delta}((z+(d/c))/i)^{k+\delta}(\det Q)^{-1/2}\Big) \times \\ \sum_{r\in R_c} e^{\pi i(a/c)Q[r]} \sum_{v\in\mathbb{Z}^n} P(Q^{-1/2}v)e^{\pi i z Q^{-1}[v]}e^{\pi i(d/c)Q^{-1}[v]}e^{2\pi i v^t r/c}. \tag{3.4}$$

We would like to interchange the sum and show that the resulting inner sum over $r \in R_c$ is independent of $v$ and then apply Poisson summation one more time. Unfortunately, this does not quite work at this point; we need one more trick, partly based on the observation that above computation holds *for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $c_\gamma \neq 0$ (indeed, we haven't yet used the assumption that $\gamma \in \Gamma_0(N)$). We write $\Theta(\gamma z) = \Theta(\gamma_0(-1/z))$, where

$$\gamma_0 := \gamma S = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c\end{pmatrix} =: \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Since $\gamma \in \Gamma_0(N)$, we have $0 \neq d = c_0$. Thus, upon replacing $(\gamma, z)$ by $(\gamma_0, (-1/z))$ in (3.4), we get

$$\begin{aligned}\Theta(\gamma z) &= \Theta(\gamma_0(-1/z)) \\ &= A_\gamma(z) \sum_{r\in R(\mathbb{Z}^n/c_0\mathbb{Z}^n)} e^{\pi i(a_0/c_0)Q[r]} \sum_{v\in\mathbb{Z}^n} P(Q^{-1/2}v)e^{\pi i(-1/z)Q^{-1}[v]}e^{\pi i(d_0/c_0)Q^{-1}[v]}e^{2\pi i v^t r/c_0} \\ &= A_\gamma(z) \sum_{v\in\mathbb{Z}^n} P(Q^{-1/2}v)e^{\pi i(-1/z)Q^{-1}[v]} \sum_{r\in R_{c_0}} e^{\pi i(a_0/c_0)Q[r]}e^{\pi i(d_0/c_0)Q^{-1}[v]}e^{2\pi i v^t r_0/c_0},\end{aligned} \tag{3.5}$$

where

$$A_\gamma(z) := c_0^{\delta}(-i)^{\delta}(((-1/z)+(d_0/c_0))/i)^{k+\delta}(\det Q)^{-1/2}.$$

As a next step, will show that the inner sums

$$\Phi_v(\gamma_0) := \sum_{r\in R_{c_0}} e_{2c_0}\left(a_0Q[r] + d_0Q^{-1}[v] + 2r^tv\right), \quad e_q(w) := e^{2\pi i w/q},$$

do *not* depend upon $v$ (then we will apply Poisson summation to the $v$-sum). Note at this point that the sum does not depend upon the set of coset representatives $R_{c_0}$; it is really a sum over $\mathbb{Z}^n/c_0\mathbb{Z}^n$. Recall also that $d_0 = -c$ is not zero and that $d_0$ is a multiple of $N$, hence $d_0Q^{-1}[v] = (d_0/N)NQ^{-1}[v] \in 2\mathbb{Z}$ for all $v \in \mathbb{Z}^n$. We want to apply a change of variables to complete the square in the finite sum $\Phi_v(\gamma_0)$, by replacing $r$ by $r + \xi$, for a suitable $\xi \in \mathbb{Z}^n/c_0\mathbb{Z}^n$. We compute

$$a_0Q[r+\xi] = a_0Q[r] + 2a_0r^tQ\xi + a_0Q[\xi].$$

Motivated by the fact that we see the term $2r^tv$ in $\Phi_v(\gamma_0)$, we take $\xi = \lambda Q^{-1}v$ here, for some $\lambda \in \mathbb{Z}$ to be determined. Then

$$a_0Q[r+\xi] = a_0Q[r] + 2a_0\lambda r^tv + a_0\lambda^2Q^{-1}[v].$$

From $1 = a_0d_0 - b_0c_0$, we obtain

$$2 = 2a_0d_0 - b_0(2c_0) \equiv 2a_0d_0 \pmod{2c_0}.$$

Consequently, if we take $\lambda = d_0$, then

$$\begin{aligned} a_0 Q[r+\xi] &= a_0 Q[r] + 2a_0 d_0 r^t v + a_0 d_0^2 Q^{-1}[v] \\ &= a_0 Q[r] + 2a_0 d_0 r^t v + (2a_0 d_0)(d_0 Q^{-1}[v]/2) \\ &\equiv a_0 Q[r] + 2r^t v + 2(d_0 Q^{-1}[v]/2) \pmod{2c_0} \\ &= a_0 Q[r] + 2r^t v + d_0 Q^{-1}[v], \end{aligned}$$

where we used that $(d_0 Q^{-1}[v]/2) \in \mathbb{Z}$! Thus, we indeed have independence of $v$:

$$\Phi_v(\gamma_0) = \sum_{r \in \mathbb{Z}^n / c_0 \mathbb{Z}^n} e_{c_0}(a_0 Q[r]/2) =: \Phi(\gamma_0).$$

As we announced, we can now "pull out" the inner finite sum in (3.5), that is, $\Phi(\gamma_0)$ and apply Poisson summation to the remaining $v$-sum, giving

$$\Theta(\gamma z) = \Phi(\gamma_0) A_\gamma(z) \sum_{v \in \mathbb{Z}^n} P(Q^{-1/2} v) e^{\pi i (-1/z) Q^{-1}[v]} = \Phi(\gamma_0) A_\gamma(z) (-i)^\delta (z/i)^{-k-\delta} (\det Q)^{1/2} \Theta(z).$$

We are almost done. We only need to simplify the factors in front. We recall that $c_0 = d$, $d_0 = -c$ and compute

$$\begin{aligned} A_\gamma(z)(-i)^\delta (z/i)^{-k-\delta} (\det Q)^{1/2} &= c_0^\delta (-i)^\delta \left( \frac{(-1/z) + d_0/c_0}{i} \right)^{k+\delta} (\det Q)^{-1/2} \left( \frac{z}{i} \right)^{k+\delta} (-i)^\delta (\det Q)^{1/2} \\ &= d^\delta (-i)^{2\delta} \left( \frac{(-1/z) + (-c/d)}{i} \frac{z}{i} \right)^{k+\delta} = d^{-k} (cz+d)^{k+\delta}, \end{aligned}$$

where we used – for the first time – that $k$ is an integer and that $\delta$ is an even integer. Thus, we have proved that

$$\Theta(\gamma z) = \left( d^{-k} \sum_{r \in \mathbb{Z}^n / d\mathbb{Z}^n} e_d(bQ[r]/2) \right) (cz+d)^{k+\delta} \Theta(z).$$

We assumed that $c \neq 0$, but note that above identity also holds if $c = 0$. Indeed, if $c = 0$, then $d \in \{\pm 1\}$ and, as noted earlier, $\Theta(\gamma z) = \Theta(z)$, while

$$d^{-k} \sum_{r \in \mathbb{Z}^n / d\mathbb{Z}^n} e_d(bQ[r]/2)(cz+d)^{k+\delta} = d^{-k} d^{k+\delta} = d^\delta = 1,$$

since $\delta$ is even. Thus, to prove the theorem, we must show that the function $\psi : \Gamma_0(N) \to \mathbb{C}$, defined by

$$\psi(\gamma) := d^{-k} \sum_{r \in \mathbb{Z}^n / d\mathbb{Z}^n} e_d(bQ[r]/2), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad (N \geq 2), \tag{3.6}$$

is a actually a group homomorphism from $\Gamma_0(N)$ to $\mathbb{C}^\times$. First, it follows from $\Theta(\gamma z) = \psi(\gamma)\mu(\gamma, z)\Theta(z)$ that $\psi(\gamma) \neq 0$ for all $\gamma \in \Gamma_0(N)$, because if we had $\psi(\gamma) = 0$ for some $\gamma$, then $\Theta(\gamma z) = 0$ for all $z \in \mathbb{H}$, hence $\Theta(\tau) = 0$ for all $\tau \in \mathbb{H}$, contradicting our hypothesis that $\Theta \neq 0$. Moreover, we have

$$\begin{aligned} \psi(\gamma_1\gamma_2)\mu(\gamma_1\gamma_2, z)^{k+\delta}\Theta(z) = \Theta(\gamma_1(\gamma_2 z)) &= \psi(\gamma_1)\mu(\gamma_1, \gamma_2 z)^{k+\delta}\Theta(\gamma_2 z) \\ &= \psi(\gamma_1)\mu(\gamma_1, \gamma_2 z)^{k+\delta}\psi(\gamma_2)\mu(\gamma_2, z)^{k+\delta}\Theta(z). \end{aligned}$$

Appealing to the cocycle relation $\mu(\gamma_1\gamma_2) = (\mu(\gamma_1) \circ \gamma_2) \cdot \mu(\gamma_2)$, we deduce

$$\psi(\gamma_1\gamma_2)\Theta(z) = \psi(\gamma_1)\psi(\gamma_2)\Theta(z).$$

As we are assuming that $\Theta$ does not vanish identically, this implies $\psi(\gamma_1\gamma_2) = \psi(\gamma_1)\psi(\gamma_2)$, as desired. Hence $\psi : \Gamma_0(N) \to \mathbb{C}^\times$ is indeed a homomorphism and this finishes the proof of Theorem 1.

3.2. **Second step: Determination of the character.** We retain the set up and the notation from the previous section. In particular, $\Theta$ abbreviates $\Theta_{Q,P}$. We now assume that $N$ is in fact equal to the *level* of $Q$, the *smallest* integer $N \geq 1$ such that $NQ^{-1}$ is even integral. As before, we assume that $N \geq 2$. It is not hard to check (using division with remainder) that for all $m \in \mathbb{Z}$ such that $mQ^{-1}$ is even integral, we have $N|m$. In particular, we have $N|2\det Q$, because $(\det Q)Q^{-1}$ has integral entries, since $Q^{-1} = (1/\det Q)Q^{\mathrm{adj}}$, where $Q^{\mathrm{adj}}$ is the adjuagte matrix whose entries are minors of $Q$ and thus integers.

We would like to know more about the character $\psi$, whose existence is guaranteed by Theorem 1 and is written down explicitly in (3.6).

We first claim that $\psi(\gamma T^m) = \psi(\gamma)$ for all $m \in \mathbb{Z}$ and all $\gamma \in \Gamma_0(N)$. To prove this claim, note that, on the one hand,

$$\begin{aligned}\Theta(\gamma T^m z) &= \psi(\gamma T^m)\mu(\gamma T^m, z)^{k+\delta}\Theta(z)\\ &= \psi(\gamma T^m)\mu(\gamma, T^m z)^{k+\delta}\mu(T^m, z)^{k+\delta}\Theta(z)\\ &= \psi(\gamma T^m)\mu(\gamma, T^m z)^{k+\delta}\Theta(z),\end{aligned} \tag{3.7}$$

since $\mu(T^m, z) = 1$. On the other hand, we have

$$\Theta(\gamma T^m z) = \psi(\gamma)\mu(\gamma, T^m z)^{k+\delta}\Theta(T^m z) = \psi(\gamma)\mu(\gamma, T^m z)^{k+\delta}\Theta(z), \tag{3.8}$$

since $\Theta(T^m z) = 1$. Subtracting (3.8) from (3.7), we get

$$0 = \left(\psi(\gamma T^m) - \psi(\gamma)\right)\mu(\gamma, T^m z)^{k+\delta}\Theta(z).$$

Choosing a point $z \in \mathbb{H}$ such that $\Theta(z) \neq 0$ (recall our standing assumption that $\Theta$ does not vanish identically), we get the claim. We have

$$\gamma T^m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+ma \\ c & d+mc \end{pmatrix}. \tag{3.9}$$

What we just proved, combined with (3.6), gives

$$\psi(\gamma) = d^{-k}\sum_{r \in \mathbb{Z}^n/d\mathbb{Z}^n} e_d(bQ[r]/2) = (d+mc)^{-k}\sum_{r \in \mathbb{Z}^n/(d+mc)\mathbb{Z}^n} e_{d+mc}((b+ma)Q[r]/2). \tag{3.10}$$

Suppose next that $c \neq 0$, and that $m \in \mathbb{Z}$ is an integer such $d + mc = p$ is an odd prime number. By Dirichlet's theorem and since $c$ and $d$ are coprime, there are infinitely primes of this form. Under this assumption,

$$\psi(\gamma) = p^{-k}\sum_{r \in \mathbb{Z}^n/p\mathbb{Z}^n} e_p(\lambda Q[r]/2), \quad \lambda = b + am. \tag{3.11}$$

Note that $\lambda$ defines an element of $\mathbb{F}_p^\times$ since $1 = ap - (b+am)c \equiv -c\lambda \pmod p$. Choose $\alpha = \alpha_p \in \mathbb{Z}$ such that $2\alpha \equiv 1 \pmod p$. Then, for all $r \in \mathbb{Z}^n$, we have $\lambda Q[r]/2 \equiv \lambda\alpha Q[r] \pmod p$ and so $e_p(\lambda Q[r]/2) = e_p(\alpha\lambda Q[r])$. The symmetric, $\mathbb{Z}$-bilinear form $(r, s) \mapsto (\lambda\alpha)r^t Qs$, $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ induces an $\mathbb{F}_p$-linear form on the $\mathbb{F}_p$-vector space $V_p := \mathbb{Z}^n/p\mathbb{Z}^n$. A standard fact about symmetric bilinear forms over fields of characteristic $\neq 2$ (non-degenerate or not) is the existence of $A \in \mathrm{Aut}_{\mathbb{F}_p}(V_p)$ and $t_i \in \mathbb{Z}$, so that

$Q[Ar] = \sum_{i=1}^n t_i r_i^2 + p\mathbb{Z}^n$ for all $r \in V_p$ and so that $\det(A)^2(t_1\cdots t_n) \equiv \det Q \pmod p$. Therefore, recalling the Gauss sums $G(a,q) = \sum_{x\in\mathbb{Z}/q\mathbb{Z}} e_q(x^2)$ and their properties from §2.4, we have,

$$\begin{aligned}\psi(\gamma) &= p^{-k}\prod_{i=1}^n G(\lambda\alpha t_i, p) = p^{-n/2}\prod_{i=1}^n \left(\frac{\lambda\alpha t_i}{p}\right) G(1,p)\\ &= p^{-n/2}G(1,p)^n\left(\frac{(\alpha\lambda)^n(t_1\cdots t_n)}{p}\right) = \varepsilon_p^n\left(\frac{\det Q}{p}\right),\end{aligned}$$

provided that $p \nmid \det Q$, so that we can use $G(\lambda t_i, p) = \left(\frac{\lambda t_i}{p}\right) G(1,p)$. Note that $(\alpha\lambda)^n \in \mathbb{F}_p^{\times 2}$ is a square, since $n$ is even. The final simplification is

$$\psi(\gamma) = \varepsilon_p^n\left(\frac{\det Q}{p}\right) = (\varepsilon_p^2)^k\left(\frac{\det Q}{p}\right) = \left(\frac{-1}{p}\right)^k\left(\frac{\det Q}{p}\right) = \left(\frac{(-1)^k\det Q}{p}\right).$$

This holds if $c \neq 0$ and for all odd prime numbers $p$ such that $p \equiv d \pmod c$ and $p \nmid \det Q$ (of which there are infinitely many). Note that this implies in particular that $\psi(\gamma) \in \{\pm 1\}$ for all $\gamma \in \Gamma_0(N)$.

We proceed with some other consequences of (3.10) and (3.11). Let us define, for any $b, d \in \mathbb{Z}$ which form the second column of matrix in $\Gamma_0(N)$, the sum

$$G_Q(b,d) := d^{-k}\sum_{r\in\mathbb{Z}^n/d\mathbb{Z}^n} e_d(bQ[r]/2).$$

So $G_Q(b,d) = \psi(\gamma)$ for $\gamma \in \Gamma_0(N)$ with $b = b_\gamma$ and $d = d_\gamma$. For any nonzero $p \in \mathbb{Z}$, not necessarily a prime number for now, let $\zeta_p \in \mathbb{C}^\times$ be a primitive $|p|$th root of unity, let $\mathbb{Q}(\zeta_p)$ denote the smallest subfield of $\mathbb{C}$ containing $\zeta_p$. Recall that that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is Galois and that $(\mathbb{Z}/p\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. An isomorphism is given by assigning to $h \in (\mathbb{Z}/p\mathbb{Z})^\times$, the unique automorphism of $\mathbb{Q}(\zeta_p)$ extending $\zeta_p \mapsto \zeta_p^h$.

Now consider an arbitrary $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$ and an arbitrary $m \in \mathbb{Z}$. Recall that (3.9) implied

$$G_Q(b,d) = G_Q(b+ma, d+mc) \in \mathbb{Q}(\zeta_{d+mc}).$$

Since $m$ is arbitrary, this gives a (second[4]) proof of the fact $G_Q(b,d) \in \mathbb{Q}$. Indeed, $d+mc$ can take two coprime values, two distinct odd primes for example, and $\mathbb{Q}(\zeta_p)\cap\mathbb{Q}(\zeta_{p'}) = \mathbb{Q}$ for coprime $p$, $p'$. Let us write, as we did above, $\lambda = b+ma$ and let $\lambda' \in \mathbb{Z}$ be such that $\lambda\lambda' \equiv 1 \pmod{d+mc}$. The automorphism of $\mathbb{Q}(\zeta_{d+mc})$, extending $\zeta_{d+mc} \mapsto \zeta_{d+mc}^{\lambda'}$ sends $G_Q(\lambda, d+mc)$ to $G_Q(1, d+mc)$. But $G_Q(\lambda, d+mc)$ is also a rational number, so applying this automorphism has no effect on this sum and hence

$$G_Q(b,d) = G_Q(b+mc, d+mc) = G_Q(1, d+mc) = G_Q(1,d).$$

This shows in particular that $\psi(\gamma)$ depends only on the lower right entry $d = d_\gamma$ of $\gamma$! Let us now write $c = Nc'$ for some $c' \in \mathbb{Z}$. Then the matrix $\gamma' = \left(\begin{smallmatrix} a & bc'\\ N & d\end{smallmatrix}\right)$ belongs again to $\Gamma_0(N)$ and has the same lower right entry as $\gamma$ and so

$$G_Q(1,d) = \psi(\gamma) = \psi(\gamma') = G_Q(bc', d) = G_Q(1, d+mN).$$

Thus we see that $\psi(\gamma)$ depends in fact only on the residue class $d_\gamma + N\mathbb{Z} \in (\mathbb{Z}/N\mathbb{Z})^\times$. In particular, since $\psi(I) = 1$ and $I$ has lower right entry 1, we get that $\psi$ is trivial on $\Gamma_1(N)$. Finally, let us recall that the assignment $\gamma \mapsto (d_\gamma + N\mathbb{Z})^\times$, gives rise to an exact sequence of groups

$$1 \to \Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 1.$$

[4] we already showed that, $G_Q(b,d) \in \{\pm 1\}$, but note that the above argument is independent of this.

Since $\psi$ is trivial on $\Gamma_1(N)$, there is thus a unique character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ such that $\psi(\gamma) = \chi(d_\gamma + N\mathbb{Z})$ for all $\gamma \in \Gamma_0(N)$.

We have proved the following theorem.

**Theorem 2.** *Let $k \geq 1$ be an integer, $n := 2k$. Let $Q \in M_n(\mathbb{Z})$ be an integral, symmetric, positive definite matrix with even diagonal entries. Let $N$ be the level of $Q$, that is, the smallest positive integer $N \geq 1$, such that $NQ^{-1}$ has integral entries with even diagonal entries. Assume that $N \geq 2$. Let $P : \mathbb{R}^n \to \mathbb{C}$ be a harmonic polynomial, which is homogeneous of even degree $\delta \geq 0$. Suppose that the theta function*

$$\Theta_{Q,P}(z) = \sum_{v \in \mathbb{Z}^n} P(Q^{1/2}v) e^{\pi i z Q[v]} = \sum_{\lambda \in L_Q} P(\lambda) e^{\pi i z |\lambda|^2}, \quad L = Q^{1/2}\mathbb{Z}^n, z \in \mathbb{H},$$

*does not vanish identically on $\mathbb{H}$. Then there is a unique character $\chi = \chi_Q : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ such that for all $\gamma \in \Gamma_0(N)$ and all $z \in \mathbb{H}$ we have*

$$\Theta_{Q,P}(\gamma z) = \chi(d_\gamma + N\mathbb{Z}) \mu(\gamma, z)^{k+\delta} \Theta_{Q,P}(z).$$

*In the notation of Theorem 1, we have $\psi(\gamma) = \chi(d_\gamma + N\mathbb{Z})$. Moreover, for all $\gamma \in \Gamma_0(N)$ with $c_\gamma \neq 0$ and all odd prime numbers $p \equiv d_\gamma \pmod{c_\gamma}$ and $p \nmid \det Q$, we have*

$$\chi(d_\gamma + N\mathbb{Z}) = \left( \frac{(-1)^k \det Q}{p} \right). \tag{3.12}$$

3.3. **Third setp: holmorphy at the cusps.** We have shown that the theta function $\Theta_{Q,P}$ *transforms like* modular form in $M_{k+\delta}(\Gamma_0(N), \chi_Q) \subseteq M_{k+\delta}(\Gamma_1(N))$. It remains to show that it is indeed a modular form. This follows from the following general principle.

**Proposition 3.1.** *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and $N \geq 1$ an integer such that $\Gamma(N) \leq \Gamma$. Let $\kappa \geq 1$ be an integer and let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function which transforms like a modular form of weight $\kappa$ for the group $\Gamma$, that is, satisfies $f \circ \gamma = \mu(\gamma)^\kappa \cdot f$ for all $\gamma \in \Gamma$. Suppose that $f$ admits a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N}$ and that there exist real numbers $C, r > 0$ such that $|a(n)| \leq C n^r$ for all $n \geq 1$. Then $f \in M_\kappa(\Gamma)$.*

*Proof.* We write $z = x + iy$. First we show that $|f(z)| \lesssim 1 + y^{-r-1}$ with implied constants depending only on $C, r, |a(1)|$ and $N$. For this, we first use the triangle inequality and the assumptions on $a(n)$, giving the uniform bound

$$|f(z)| \leq |a(1)| + \sum_{n=1}^{\infty} |a(n)| e^{-2\pi y/N} \lesssim 1 + \sum_{n=1}^{\infty} n^r e^{-2\pi y/N}.$$

Next we define $C_r := \sup_{t>0} t^r e^{-t}$ and use that for all $n \geq 1$ and all $y > 0$,

$$n^r e^{-2\pi n y/N} = \left( n^r e^{-\pi n y/N} \right) e^{-\pi n y/N} = (\pi y/N)^{-r} \left( (\pi n y/N)^r e^{-\pi n y/N} \right) \leq C_r y^{-r} (\pi/N)^{-r} e^{-\pi n y/N}.$$

The remaining geometric series is

$$\sum_{n=1}^{\infty} e^{-\pi n y/N} = \frac{e^{-\pi y/N}}{1 - e^{-\pi y/N}} = \frac{1}{e^{\pi y/N} - 1} \sim \frac{N}{\pi y} \qquad \text{as } y \to 0.$$

This proves our first estimate $|f(z)| \lesssim y^{-(r+1)}$ for $y \leq 1$. Now fix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$. Then the function $g := g_\gamma := \mu(\gamma)^{-\kappa}(f \circ \gamma)$ transforms like a modular form of weight $\kappa$ for the group

$\gamma^{-1}\Gamma\gamma \supset \Gamma(N)$. In particular, it is $N$-periodic and admits a Fourier expansion

$$g(\tau) = \sum_{n\in\mathbb{Z}} c_g(n)e^{2\pi in\tau/N}, \quad c_g(n) = \frac{1}{N}\int_{iy}^{iy+N} g(z)e^{-2\pi inz/N}dz.$$

We need to show that $c_g(n) = 0$ if $n < 0$ (because this means, by definition, that $f$ is holomorphic at the cusp $\gamma\infty$). The triangle inequality shows that for all $y > 0$ and all $n \in \mathbb{Z}$,

$$|c_g(n)| \leq \frac{1}{N} \sup_{x\in[0,N]} |g(iy + x)|e^{2\pi ny/N}.$$

From here we see that it suffices find $A \in \mathbb{R}$, so that $\sup_{x\in[0,N]} |g(iy + x)| = O(y^A)$, as $y \to \infty$. Indeed, for $n < 0$, this estimate for $|c_g(n)|$, will imply $c_g(n) = 0$ as we can then let $y \to \infty$. As $c \neq 0$, we have

$$\mathrm{Im}(\gamma z) = \frac{y}{|cz + d|^2} = \frac{y}{(cy)^2 + (cx + d)^2} \leq \frac{1}{y},$$

so that $\mathrm{Im}(\gamma z) \to 0$ as $y \to \infty$. From what we showed at the beginning of the proof, we have, as $y \to \infty$ and $x \in [0, N]$,

$$\begin{aligned}|g(z)| &\lesssim \mathrm{Im}(\gamma z)^{-(r+1)} = |cz + d|^{2r+2}y^{-(r+1)} \\ &= c^{2r+2}(y^2 + (x + d/c)^2)^{r+1}y^{-(r+1)} \lesssim_{c,d,N,r} y^{r+1}.\end{aligned}$$

So what we want holds for $A = r + 1$. □

We may apply Proposition 3.1 to $\Gamma = \Gamma_1(N)$, $\kappa = k + \delta$ and $f = \Theta_{Q,P}$, since we have

$$\Theta(z) = \sum_{n=0}^{\infty} c_{P,Q}(n)e^{2\pi inz}, \quad c_{P,Q}(n) = \sum_{\lambda\in L_Q:|\lambda|^2=2n} P(\lambda),$$

where $L_Q := Q^{1/2}\mathbb{Z}^n \subseteq \mathbb{R}^n$ and the coefficients $c_{P,Q}(n)$ clearly grow at most polynomially. Thus we have proved

**Theorem 3.** *In addition to Theorem 2, we have* $\Theta_{Q,P} \in M_{k+\delta}(\Gamma_0(N), \chi_Q) \subseteq M_{k+\delta}(\Gamma_1(N))$.

## 4. Vector valued theta functions

Here, we present a different approach to study the modularity properties of theta functions, by introducing a representation of $\mathrm{SL}_2(\mathbb{Z})$ on the space of functions on the discriminant group of the lattice. This representation encodes the transformation property of a vector-valued theta function, whose components are shifted theta functions. The idea is then to show that this representation is trivial on a congruence subgroup, which proves the modualirty of *all* shifted theta functions at once. While this section is almost independent of the others, it does not reprove Theorem 3, it merely offers a different point of view. I hope to complete the discussion and give an alternative proof of Theorem 3 using the objects introduced here at some point.

Let $L \subseteq \mathbb{R}^n$ be an even lattice with dual lattice $L^* \subseteq \mathbb{R}^n$. We will assume in this section that $L \subseteq \mathbb{Q}^n$. The latter assumption will only be used in the proof of Theorem 4 and the discussion applies more generally whenever the statement of Theorem 4 holds for the lattice under consideration (In certain examples of interest, this is not hard to verify). Let $P : \mathbb{R}^n \to \mathbb{C}$ be a harmonic polynomial of degree $\delta \geq 0$. We assume that $\delta$ is even and consider, as before, the theta function $\Theta_{L,P}(z) = \sum_{v\in L} P(v)e^{\pi i|v|^2 z}$. For each $x \in \mathbb{R}^n$, we define the *shifted theta function* as

$$\Theta_{x+L,P}(z) := \sum_{v\in x+L} P(v)e^{\pi i|v|^2 z} = \sum_{v\in L} P(v+x)e^{\pi i|v+x|^2 z}.$$

Since $\delta$ is even, we see that $\Theta_{x+L,P}(z) = \Theta_{-x+L,P}(z)$ and we clearly have $\Theta_{x+L,P} = \Theta_{L,P}$ for all $x \in L$. In the following, we regard $L$ and $P$ as fixed and don't always display these parameters in our notation.

The *discriminant group* of $L$ is defined as $D := L^*/L$. It is a finite abelian group of order $|L|^2$. We write $L^2(D)$ for the $|D|$-dimensional complex Hilbert space consisting of all functions $f : D \to \mathbb{C}$, equipped with the Hermitian inner product

$$\langle \phi_1, \phi_2\rangle_{L^2(D)} := \frac{1}{|D|}\sum_{x\in D} \phi_1(x)\overline{\phi_2(x)}.$$

The finite abelian group $D$ inherits from $\mathbb{R}^n$ a non-degenerate symmetric, $\mathbb{Z}$-bilinear form $D \times D \to \mathbb{R}/\mathbb{Z}$, $\langle x + L, y + L\rangle := \langle x, y\rangle_{\mathbb{R}^n} + \mathbb{Z}$, for $x, y \in L^*$. Using it, we define the Fourier transform $F : L^2(D) \to L^2(D)$ by

$$F\phi(x) := \frac{1}{|D|^{1/2}}\sum_{y\in D} \phi(y)e(-\langle x, y\rangle), \quad x \in D, \phi \in L^2(D), \quad e(r) = e^{2\pi ir}, r \in \mathbb{R}/\mathbb{Z}.$$

For later use, let us also define, for all $a, b \in \mathbb{Z}$, with $\gcd(|D|, a) = 1$, the unitary operators $Q_b, R_a : L^2(D) \to L^2(D)$ by

$$Q_b\phi(x) := e((b/2)|x|^2)\phi(x), \quad R_a\phi(x) = \phi(ax).$$

We will abbreviate $Q := Q_1$, so that $Q_b = Q^b$ for all $b \in \mathbb{Z}$. Finally, we we define the $L^2(D)$-valued function $\vec{\Theta} = \vec{\Theta}_{L,P} : \mathbb{H} \to L^2(D)$ by

$$\vec{\Theta}(z)(x) := \Theta_{x+L,P}(z), \quad \text{for } x \in D, z \in \mathbb{H}.$$

For $x \in D$ we write $\mathbf{e}_x \in L^2(D)$ for the function $\mathbf{e}_x(y) = \delta(x, y)$ (Kronecker delta), so that we can also write

$$\vec{\Theta}(z) = \sum_{y\in D} \Theta_{y+L,P}(z)\mathbf{e}_y.$$

We will see that the operators $Q = Q_1$ and $F$ describe how $\vec{\Theta}$ transforms under the action of the generators $S, T \in \mathrm{SL}_2(\mathbb{Z})$. To see how $T$ acts, we note that for all $x \in L^*, v \in L$ $z \in \mathbb{H}$ we have

$$e^{\pi i|x+v|^2(z+1)} = e^{\pi i|x+v|^2 z}e^{\pi i\left(|x|^2+2\langle x,v\rangle+|v|^2\right)} = e^{\pi i|x|^2}e^{\pi i|x+v|^2 z}.$$

This implies $\Theta_{x+L,P}(z+1) = e^{\pi i|x|^2}\Theta_{x+L,P}(z)$ and $\vec{\Theta}(z+1) = Q\vec{\Theta}(z)$ for all $z \in \mathbb{H}$. To see how $S$ acts, we compute, for each $x \in L^*/L$, using Poisson summation (Corollary 2.1),

$$\begin{aligned}\Theta_{x+L,P}(-1/z) &= \sum_{v\in L} P(x+v)e^{\pi i|x+v|^2(-1/z)} \\ &= (z/i)^{n/2+\delta} i^{-\delta} \frac{1}{|L|} \sum_{w\in L^*} P(w)e^{\pi iz|w|^2} e^{-2\pi i\langle x,w\rangle} \\ &= (z/i)^{n/2+\delta} i^{-\delta} \frac{1}{|L|} \sum_{y\in L^*/L} \sum_{v\in L} P(y+v)e^{\pi iz|y+v|^2} e^{-2\pi i\langle x,y+v\rangle} \\ &= (z/i)^{n/2+\delta} i^{-\delta} \frac{1}{|L|} \sum_{y\in L^*/L} \Theta_{y+L,P}(z)e^{-2\pi i\langle y,x\rangle}.\end{aligned}$$

Here, in the second step, our normalization of the Fourier transform and Poisson summation would more naturally give the factor $e^{2\pi i\langle x,w\rangle}$ instead of $e^{-2\pi i\langle x,w\rangle}$, but by changing variables $w$ to $-w$ and using that $\delta$ is even, we get the above form. Since $|L|^2 = |D|$, this shows that

$$\vec{\Theta}(-1/z) = i^{-\delta}(z/i)^{n/2+\delta} F(\vec{\Theta}(z)).$$

Here, $n$ could be even or odd (in which case $(z/i)^{n/2}$ is defined as explained after Lemma 2.1). *From now on, we assume that $n = 2k$, $k \in \mathbb{Z}_{\geq 1}$ is an even integer.* Again, since $\delta$ is even, we can then write the above as

$$\vec{\Theta}(-1/z) = z^{k+\delta} i^{-k} F(\vec{\Theta}(z)).$$

We now ask whether the assignments $S \mapsto i^{-k}F$ and $T \mapsto Q = Q_1$ extend to a group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to U(L^2(D))$ (= group of unitary operators on $L^2(D)$), which might allow us to describe the action of all of $\mathrm{SL}_2(\mathbb{Z})$ on $\vec{\Theta}$. To see whether this is true, we need the (not so well-known) fact that the relations $S^2 = (ST)^3$ and $S^4 = I$ are defining relations for $\mathrm{SL}_2(\mathbb{Z})$, meaning that the group $\mathrm{SL}_2(\mathbb{Z})$ could be abstractly defined as $\langle s,t \,|\, s^4 = 1, (st)^3 = s^2\rangle$. (In contrast, $\mathrm{PSL}_2(\mathbb{Z}) \cong \langle s,t \,:\, s^2 = 1 = (st)^3\rangle$); we already mentioned these facts in §2.3. The relation $S^4 = I$ is not a problem, because, $(i^m F)^4 = F^4 = \mathrm{id}_{L^2(D)}$ for all $m \in \mathbb{Z}$, by Fourier inversion. For the relation $(ST)^3 = S^2$ we need to do a computation.

**Lemma 4.1.** *In $U(L^2(D))$ we have $(FQ)^3 = \gamma_D F^2$, where $\gamma_D := \frac{1}{|D|^{1/2}}\sum_{x\in D} e^{\pi i|x|^2}$.*

*Proof.* Given $x, y \in D$ we write, in this proof, for typographical convenience, simply $xy = \langle x,y\rangle$, $x^2 = \langle x,x\rangle$. By Fourier inversion, we need to show that

$$FQFQFQ\phi(x) = \gamma_D\phi(-x)$$

holds for all $x \in D$ and all $\phi \in L^2(D)$. By linearity, it suffices to prove this for all functions $\phi = \mathbf{e}_x$. We fix $x_0 \in D$ and write $\phi_0 := \mathbf{e}_{x_0}$, so $\phi_0(x) = 1$ if $x \neq x_0$ and $\phi_0(x) = 0$ otherwise. Naturally, the computation will proceed in six steps. Step 1:

$$Q\phi_0(x) = \phi_0(x)e(x_0^2/2).$$

Step 2:

$$FQ\phi_0(x) = e(x_0^2/2)F\phi_0(x) = e(x_0^2/2)\frac{1}{|D|^{1/2}}e(-xx_0).$$

Step 3:

$$QFQ\phi_0(x) = e(x_0^2/2)\frac{1}{|D|^{1/2}}e(-xx_0)e(x^2/2) = \frac{1}{|D|^{1/2}}e((1/2)(x-x_0)^2).$$

Step 4:

$$\begin{aligned} FQFQ\phi_0(x) &= \frac{1}{|D|}\sum_{y\in D} e((1/2)(y-x_0)^2)e(-yx) \\ &= \frac{1}{|D|}e(-xx_0)\sum_{y\in D} e((1/2)(y-x_0)^2)e(-(y-x_0)x) \\ &= \frac{1}{|D|}e(-xx_0)\sum_{z\in D} e((1/2)z^2)e(-zx) \\ &= \frac{1}{|D|}e(-xx_0)\sum_{z\in D} e((1/2)(z^2-2zx)) \\ &= \frac{1}{|D|}e(-xx_0)\sum_{z\in D} e((1/2)(z^2-2zx+x^2-x^2)) \\ &= \frac{1}{|D|^{1/2}}e(-xx_0)e(-x^2/2)\gamma_D, \end{aligned}$$

Step 5:

$$QFQFQ\phi_0(x) = \gamma_D\frac{1}{|D|^{1/2}}e(-xx_0)e(-x^2/2)e(x^2/2) = \gamma_D\frac{1}{|D|^{1/2}}e(-xx_0).$$

Step 6:

$$FQFQFQ\phi(x) = \gamma_D\frac{1}{|D|^{1/2}}\frac{1}{|D|^{1/2}}\sum_{y\in D} e(-yx_0)e(-xy) = \gamma_D\frac{1}{|D|}\sum_{y\in D} e(-y(x+x_0)) = \gamma_D\phi_0(-x).$$

This proves what we wanted. □

The Lemma shows that, for any fourth root of unity $\zeta \in \mathbb{C}^{(1)}$, we have

$$((\zeta F)Q)^3 = \zeta^3(FQ)^3 = \zeta^3\gamma_D F^2 = \zeta\gamma_D(\zeta F)^2. \tag{4.1}$$

This implies that $S \mapsto \zeta F$, $T \mapsto Q$ extends to a homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to U(L^2(D))$ if and only if $\zeta\gamma_D = 1$. Remarkably, $\gamma_D$ does not depend on the lattice $L$, but only on the dimension $n$, as the next theorem will show. For its proof we start similarly as in Appendix 4 to [1], but can then simplify the end, as we only consider positive definite lattices. The assumption that $n$ is even is *not* used in the proof given below.

**Theorem 4.** *In the above notation, we have $\gamma_D = e^{2\pi i n/8}$.*

*Proof.* Let us recall that $\gamma_D = \gamma(L) := \frac{1}{|L|}\sum_{v\in L^*/L} e^{\pi i|v|^2}$. We pause to note that $\gamma(L)$ is well-defined *for all* even lattices $L \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$. Suppose we have two such even lattices $L_1, L_2 \subseteq \mathbb{Q}^n$. Then the intersection $L_1 \cap L_2$ is also an even lattice in $\mathbb{Q}^n$ and of finite index in $L_1$ and in $L_2$. This is so because we can find $M \in \mathbb{Z}_{\geq 1}$, a common multiple of all denominators of all coordinates of some chosen basis vectors for $L_1$ and $L_2$, such that $M\mathbb{Z}^n \subseteq L_1 \cap L_2 \subseteq (1/M)\mathbb{Z}^n$, proving that $L_1 \cap L_2$ is a lattice in $\mathbb{Q}^n$.[5]

Now consider an arbitrary sub-lattice $\Lambda \subseteq L$ of finite index. We claim that $\gamma(L) = \gamma(\Lambda)$. To prove the claim, we note the inclusions

$$\Lambda \subseteq L \subseteq L^* \subseteq \Lambda^*.$$

[5]The intersection of two general lattices in $\mathbb{R}^n$ need not be a lattice (of full rank), as the example $\mathbb{Z} \cap \sqrt{3}\mathbb{Z} = \{0\}$ shows.

Let $X \subseteq \Lambda^*$ be a complete set of coset representatives of $\Lambda^*/L$ and let $Y \subseteq L$ be a complete set of coset representatives for $L/\Lambda$, so that $\Lambda^* = \sqcup_{x\in X}(x+L)$ and $L = \sqcup_{y\in L}(y+\Lambda)$. Then

$$|\Lambda|\gamma(\Lambda) = \sum_{x\in X}\sum_{y\in Y} e^{\pi i|x+y|^2} = \sum_{x\in X} e^{\pi i|x|^2} \sum_{y\in Y} e^{2\pi i\langle x,y\rangle} = \sum_{x\in X} e^{\pi i|x|^2} \sum_{\ell\in L/\Lambda} e^{2\pi i\langle \ell,x\rangle},$$

where we used that $|y|^2 \in 2\mathbb{Z}$ for all $y \in Y$. For each fixed $x \in X \subseteq \Lambda^*$, the character $\ell \mapsto e^{2\pi i\langle \ell,x\rangle}$ is trivial on $L/\Lambda$, if and only if $x \in L^*$ and so

$$\gamma(\Lambda) = \frac{|L/\Lambda|}{|\Lambda|} \sum_{x\in X\cap L^*} e^{\pi i|x|^2}.$$

So far, $X$ was arbitrary, let us now write $X = \{a+b\}_{(a,b)\in A\times B}$, where $A \subseteq \Lambda^*$ and $B \subseteq L^*$ are chosen so that $0 \in A$, $\Lambda^* = \sqcup_{a\in A}(a+L^*)$ and $L^* = \sqcup_{b\in B}(b+L)$. Then $a+b \in L^* \Leftrightarrow a = 0$ and hence

$$\gamma(\Lambda) = \frac{|L/\Lambda|}{|\Lambda|} \sum_{b\in B} e^{\pi i|b|^2} = \frac{|L/\Lambda||L|}{|\Lambda|}\gamma(L) = \gamma(L),$$

since the ratio of covolumes $|\Lambda|/|L| = [L : \Lambda] = |L/\Lambda|$. This finishes the proof of the claim.

By what we have shown so far, it suffices to evaluate $\gamma(L)$ in the case where $L = 2\mathbb{Z}^n$ (for example), which we will do now. We have $|L| = 2^n$ and $L^* = (1/2)\mathbb{Z}^n$, so $D = (1/2)\mathbb{Z}^n/2\mathbb{Z}^n$. Letting $e_i \in \mathbb{Z}^n$ denote the $i$th standard basis vector, we readily check that

$$(\mathbb{Z}/4\mathbb{Z})^n \longrightarrow D, \quad (x_1+4\mathbb{Z},\dots,x_n+4\mathbb{Z}) \longmapsto \tfrac{1}{2}\sum_{i=1}^n x_i e_i + L, \quad x_i \in \mathbb{Z},$$

is a well-defined isomorphism of finite abelian groups. It follows that

$$\gamma(2\mathbb{Z}^n) = 2^{-n}\prod_{j=1}^n \sum_{r_j\in\mathbb{Z}/4\mathbb{Z}} e^{\pi i r_j^2/4} = \prod_{j=1}^n (1/2)\left(1+e^{\pi i/4}+e^{\pi i}+e^{\pi i/4}\right) = \prod_{j=1}^n \left(e^{\pi i/4}\right) = e^{\pi i n/4},$$

as desired. □

Using Theorem 4 and Lemma 4.1 we can now prove the following theorem.

**Theorem 5.** *Let the notations be as above. In particular, $k \geq 1$ is an integer, $n = 2k$, $L \subseteq \mathbb{Q}^n$ is an even lattice with discriminant group $D = L^*/L$. There is a unique unitary representation $\rho = \rho_L : \mathrm{SL}_2(\mathbb{Z}) \to U(L^2(D))$ such that*

$$\rho(S) = i^{-k}F, \quad \text{and} \quad \rho(T) = Q.$$

*Furthermore, for every harmonic polynomial $P : \mathbb{R}^n \to \mathbb{C}$ of even degree $\delta \geq 0$, the $L^2(D)$-valued function $\vec{\Theta} = \vec{\Theta}_{L,P} : \mathbb{H} \to L^2(D)$, defined as above, satisfies*

$$\vec{\Theta}(\gamma z) = \mu(\gamma,z)^{k+\delta}\rho(\gamma)\vec{\Theta}(z) \tag{4.2}$$

*for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$.*

*Proof.* As discussed after the proof of Lemma 4.1, if $\zeta \in \mathbb{C}^\times$ is a fourth root of unity, then $T \mapsto Q$, $S \mapsto \zeta F$ extends (uniquely) to a morphism $\mathrm{SL}_2(\mathbb{Z}) \to U(L^2(D))$, if and only if $\zeta\gamma_D = 1$. Since $n = 2k$ is even, we have $\gamma_D = i^k$, by Theorem 4. Thus, taking $\zeta = i^{-k}$ achieves what we want. For the second assertion, note that the representation $\rho$ allows us to define a right- action of $\mathrm{SL}_2(\mathbb{Z})$ on the space of all functions $\vec{\vartheta} : \mathbb{H} \to L^2(D)$ by $(\vec{\vartheta}|\gamma)(z) := \mu(\gamma,z)^{-k-\delta}\rho(\gamma)\vec{\vartheta}(z)$. Then $\vec{\vartheta}$ is fixed by this action, if and only if it is fixed by $S$ and $T$. We can now apply this to $\vec{\vartheta} = \vec{\Theta}$. □

**Remark 4.1.** The composition of the representation $\rho$ in Theorem 5 with the natural map $U(L^2(D)) \to U(L^2(D))/\mathbb{C}^{(1)}$ is a projective Weil representation, whose existence (as a projective representation) could also be derived abstractly from the finite Stone von Neumann Theorem on $L^2(D)$. The latter yields a projective representation $\omega : \mathrm{Sp}(D) \to L^2(D)/\mathbb{C}^{(1)}$ where $\mathrm{Sp}(D)$ is the *symplectic* group of $D$ defined as the group automorphisms of $D \times D$ preserving an alternating $\mathbb{C}^{(1)}$-valued form. There is a natural map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Sp}(D)$ and the composition with $\omega$ agrees with the composition of $\rho$ with the map into $U(L^2(D))/\mathbb{C}^{(1)}$.

Let $\rho = \rho_L$ be as in Theorem 5. We can attempt to use the theorem to prove the modularity of *all* shifted theta functions $\Theta_{x+L,P}$ simultaneously by proving that $\rho|_{\Gamma(N)}$ is trivial for some $N$. Let us give an indication why a statement of this type should be true. Suppose that $N \geq 1$ is equal to the level of $L$. In particular, $N^{1/2}L^*$ is an even lattice and so $N\langle x, y\rangle = \langle N^{1/2}x, N^{1/2}y\rangle \in \mathbb{Z}$ for all $x, y \in L^*$ and $N\langle x, x\rangle = \langle N^{1/2}x, N^{1/2}x\rangle \in 2\mathbb{Z}$ for all $x \in L^*$. It immediately follows that $Q^N = \rho_L(T^N) = \mathrm{id}_{L^2(D)}$ and therefore that

$$T^{\ell N} = \begin{pmatrix} 1 & \ell N \\ 0 & 1 \end{pmatrix}, \qquad ST^{\ell N}S^{-1} = \begin{pmatrix} 1 & 0 \\ -\ell N & 1 \end{pmatrix} \in \ker(\rho_L),$$

for all $\ell \in \mathbb{Z}$, using simply that the kernel is a normal subgroup for the other membership. If $N = 2$, then we know that the above matrices together with $-I$ generate the group $\Gamma(2)$, for example.

As a final, related remark, we mention that a result by Rademacher [3] gives a list of generators for the congruence subgroups $\Gamma_0(p)$ for primes $p \geq 3$ (and other subgroups), which is also useful for establishing modularity of theta functions.

## 5. Even self-dual lattices

For completeness, we discuss here the (easier) case of the modularity of $\Theta_L$ for *even self-dual* lattices $L \subseteq \mathbb{R}^n$, which is not covered by Theorems 1, 2, 3.

**Theorem 6.** *Let $L \subseteq \mathbb{R}^n$ be an even self-dual lattice. Then $8|n$ and $\Theta_L(z) = \sum_{v \in L} e^{2\pi i |v|^2 z}$ is a modular form of weight $n/2$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.*

*Proof.* The Poisson summation formula and $L^* = L$ imply that $\Theta_L(-1/z) = (z/i)^{n/2}\Theta_L(z)$ for all $z \in \mathbb{H}$. Since $L$ is even, we also have $\Theta_L(z+1) = \Theta_L(z)$ for all $z \in \mathbb{H}$. Thus, *if* $8|n$, then, since $(z/i)^{n/2} = z^{n/2}$ and since $S$ and $T$ generate $\mathrm{SL}_2(\mathbb{Z})$, we have $\Theta_L \in M_{n/2}(\mathrm{SL}_2(\mathbb{Z}))$. The more interesting part is to show we necessarily have $8|n$. Using $TSTSTSz = z$, we deduce, using the two transformation rules under $z \mapsto Sz$, $z \mapsto Tz$, repeatedly, that

$$\Theta_L(z) = ((TSTSz)/i)^{n/2}((TSz)/i)^{n/2}(z/i)^{n/2}\Theta_L(z).$$

Therefore, in an open subset of $z \in \mathbb{H}$, containing $i(0, +\infty)$, we have

$$((TSTSz)/i)^{n/2}((TSz)/i)^{n/2}(z/i)^{n/2} = 1. \tag{5.1}$$

By squaring this, we get

$$i^{3n} = z^n(TSz)^n(TSTSz)^n = \left(z\,(1 + (-1/z))\left(1 + \frac{-1}{1 + (-1/z)}\right)\right)^n = (z(-1/z))^n = (-1)^n = i^{2n},$$

which implies that $n = 4m$ for some $m \in \mathbb{N}$. It follows that $n/2 = 2m$ and by inserting this back into (5.1) we obtain

$$i^{6m} = z^{2m}(TSz)^{2m}(TSTSz)^{2m} = (-1)^{2m} = 1,$$

which implies $m = 2\ell$ is even and hence $n = 4m = 8\ell$, as desired. ∎

## References


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