Modular forms and applications
Exercise Sheet 1

In this exercise sheet we introduce the hyperbolic plane

$$\mathbb{H} := \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$$

and discuss basic properties we will need during the course.

Exercise 1 (Automorphisms of the upper half plane, **graded**). Given a nonempty open subset $\Omega \subseteq \mathbb{C}$, we denote by $\mathrm{Aut}(\Omega)$ the group of holomorphic automorphims of Ω.

The goal of this exercise is to establish an explicit isomorphism between the groups $\mathrm{Aut}(\mathbb{H})$ and $\mathrm{PSL}_2(\mathbb{R})$. You may deduce this from parts (a) to (f) below.

(a) Show that the map $\Psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$, $g \mapsto \Psi_g$, given by

$$\Psi_g(z) = \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ and } z \in \mathbb{H}$$

is a well-defined morphism of groups and find its kernel.

(b) Part (a) defines a group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. Show that it is transitive by showing that the orbit of the point i is all of $\mathbb{H}$.

(c) Let $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ denote the open unit disc. Let $\tau \in \mathbb{H}$ be a point. Consider the fractional linear transformation $\delta_\tau(z) = \frac{z-\tau}{z-\overline{\tau}}$. Show that δ_τ defines a holomorphic isomorphism $\delta_\tau : \mathbb{H} \to \mathbb{D}$.

(d) Given any $\varphi \in \mathrm{Aut}(\mathbb{H})$, construct a suitable $\tilde{\varphi} \in \mathrm{Aut}(\mathbb{D})$ such that $\tilde{\varphi}(0) = 0$.

(e) Generally, given two non-empty open subsets $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ and a holomorphic isomorphism $\delta : \Omega_1 \to \Omega_2$, find an isomorphism of groups $\mathrm{Aut}(\Omega_1) \cong \mathrm{Aut}(\Omega_2)$ depending upon δ.

Recall the *Schwarz Lemma* from complex analysis. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function satisfying $f(0) = 0$. Then we have $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, the following are equivalent:

- $|f|$ has a further fixed point, that is, there is $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$.
- f is a rotation around 0, that is, there is $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$.

(f) Deduce from the previous parts that the map in part (a) induces the desired isomorphism.

Solution. (a) First notice that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{P}^1(\mathbb{C})$ via linear maps (classical matrix multiplication). Moreover $\mathrm{SL}_2(\mathbb{R})$ acts on the subset $\{[z : 1], z \in \mathbb{C}\} \subset \mathbb{P}^1(\mathbb{C})$ and under the identification $\mathbb{C} \leftrightarrow \{[z : 1], z \in \mathbb{C}\}$ this is exactly the Ψ action. It remains to argue that the map $\Psi(g)$ is indeed holomorphic (over $\mathbb{H}$) and that it maps $\mathbb{H} \subset \mathbb{C}$ to itself. It is holomorphic because products of holomorphic maps are holomorphic (notice $cz+d \neq 0$ for all $z \in \mathbb{H}$). For the group actions it remains to show that $\mathrm{Im}(\Psi_g(z)) > 0$ for all $z \in \mathbb{H}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in H$, then

$$\begin{aligned}\mathrm{Im}(\Psi_g(z)) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\right) \\ &= \mathrm{Im}\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right) \\ &= \frac{(ad-bc)\,\mathrm{Im}(z)}{|cz+d|^2} \\ &= \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0.\end{aligned}$$

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ be so that Ψ_g acts trivially on $\mathbb{H}$. The equations $\Phi_g(i) = \frac{ai+b}{ci+d} = i$ implies $a = d$ and $b = -c$. From $\Psi_g(2i) = \frac{a(2i)+b}{c(2i)+d} = 2i$ we see that $b = -4c$ and so combining the two we deduce $b = c = 0$. From the determinant condition we see that $a = d = a^{-1}$. Hence $g \in \{\pm I_2\}$.

(b) Let $z = x + iy \in \mathbb{H}$, $x, y \in \mathbb{R}$, $y > 0$. Then

$$\begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \cdot i = \frac{\sqrt{y}i + \sqrt{y}^{-1}x}{\sqrt{y}^{-1}} = x + iy.$$

(c) It is clear that δ_τ is holomorphic. We first show that it takes values in $\mathbb{D}$. Let $z \in \mathbb{H}$, then since $\mathrm{Im}(z), \mathrm{Im}(\tau) > 0$ it holds that

$$\begin{aligned}(\mathrm{Im}(z) - \mathrm{Im}(\tau))^2 &= \mathrm{Im}(z)^2 - 2\,\mathrm{Im}(z)\,\mathrm{Im}(\tau) + \mathrm{Im}(\tau)^2 \\ &< \mathrm{Im}(z)^2 + 2\,\mathrm{Im}(z)\,\mathrm{Im}(\tau) + \mathrm{Im}(\tau)^2 \\ &= (\mathrm{Im}(z) + \mathrm{Im}(\tau))^2.\end{aligned}$$

and in particular we deduce that $|z - \tau| < |z - \overline{\tau}|$. Next we construct an inverse of δ_τ. One way to deduce it is to use the action on $\mathbb{P}^1(\mathbb{C})$. δ_τ can be seen as the restriction to $\mathbb{H} \subset \mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ of the linear map

$$\begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix}$$

acting on $\mathbb{P}^1(\mathbb{C})$. In particular the inverse must be represented by the linear map

$$\frac{1}{2i\,\mathrm{Im}(\tau)} \begin{pmatrix} -\overline{\tau} & \tau \\ -1 & 1 \end{pmatrix}$$

and restricting to $\mathbb{D}$ we get:

$$\delta_\tau^{-1} \colon \mathbb{D} \to \mathbb{H};\, z \mapsto \frac{\tau - \overline{\tau}z}{1 - z}.$$

(d) Let $\varphi \in \mathrm{Aut}(\mathbb{H})$ let $\tau = \varphi(i)$. Let $\tilde{\varphi} = \delta_\tau \circ \varphi \circ \delta_i^{-1}$, then it is in $\mathrm{Aut}(\mathbb{D})$ and $\tilde{\varphi}(0) = 0$.

(e) Conjugation by δ does the job.

(f) Let $\varphi \in \mathrm{Aut}(\mathbb{H})$. After, if necessary, compose by Ψ_g for some $g \in \mathrm{SL}_2(\mathbb{R})$ we may assume that $\varphi(i) = i$. Let $\tilde{\varphi} \in \mathrm{Aut}(\mathbb{D})$ as in point (d) so that $\varphi(0) = 0$. Then by Schwartz' Lemma we have for any $z \in \mathbb{D}$:

$$|z| = |\tilde{\varphi}^{-1}(\tilde{\varphi}(z))| \leq |\tilde{\varphi}(z)| \leq |z|$$

and so, always by Schwartz' Lemma, there exists $\theta \in \mathbb{R}$ so that for all $z \in \mathbb{D}$ we have $\tilde{\varphi}(z) = e^{i\theta} z$. In particular we have for all $z \in \mathbb{H}$:

$$\begin{aligned}\varphi(z) &= \delta_i^{-1} \circ \tilde{\varphi} \circ \delta_{\varphi(i)}(z) \\ &= \frac{i + ie^{i\theta}\frac{z-i}{z+i}}{1 - e^{i\theta}\frac{z-i}{z+i}} \\ &= \frac{z\frac{e^{i\theta/2}+e^{-i\theta/2}}{2} + \frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}}{z\frac{e^{-i\theta/2}-e^{i\theta/2}}{2i} + \frac{e^{-i\theta/2}+e^{i\theta/2}}{2}} \\ &= \Psi_{k(\theta/2)}(z),\end{aligned}$$

where $k(\theta/2) = \left(\begin{smallmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{smallmatrix}\right)$. In particular we get the desired Isomorphism.

Exercise 2 (Factor of automorphy)**.** For each $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we define a function

$$j(g) : \mathbb{H} \to \mathbb{C}^\times \quad \text{by} \quad j(g)(z) := j(g, z) := cz + d \ \text{ for } z \in \mathbb{H}.$$

(a) Verify that for all $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$ we have

$$j(g_1 g_2) = (j(g_1) \circ \Psi_{g_2}) \cdot j(g_2).$$

(Here, Ψ_g is as in Exercise 1)

(b) Verify that for all $g \in \mathrm{SL}_2(\mathbb{R})$ we have $(\Psi_g)' = j(g)^{-2}$. How is part (a) related to the chain-rule in complex analysis?

(c) For every integer $k \in \mathbb{Z}$, every function $F \colon \mathbb{H} \to \mathbb{C}$ and every $g \in \mathrm{SL}_2(\mathbb{R})$ we define a new function

$$F|_k g : \mathbb{H} \to \mathbb{C} \quad \text{by} \quad F|_k g := j(g)^{-k}(F \circ \Psi_g),$$

called "F slash g". Show that this defines an action of $\mathrm{SL}_2(\mathbb{R})$ on the space of all functions $F : \mathbb{H} \to \mathbb{C}$ (a right group action by linear maps). We call it the "slash-action (in weight k)".

Solution. (a) Write

$$g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}), i = 1, 2.$$

Then

$$j(g_1 g_2)(z) = (c_1 a_2 + d_1 c_2) z + c_1 b_2 + d_1 d_2$$

On the other hand

$$\begin{aligned}(j(g_1) \circ \Psi_{g_2} \cdot j(g_2))(z) &= (c_1 \frac{a_2 z + b_2}{c_2 z + d_2}) + d_1)(c_2 z + d_2) \\ &= (c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)\end{aligned}$$

(b) For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have

$$\Psi_g'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{1}{(cz + d)^2} = j(g)^{-2}(z)$$

(c)

$$\begin{aligned} F|_k(g_1g_2)(z) &= \frac{1}{j(g_1g_2)(z)^k} \cdot F \circ \Psi_{g_1g_2}(z) \\ &= \frac{1}{(j(g_1) \circ \Psi_{g_2}(z))^k (j(g_2)(z))^k} (F \circ \Psi_{g_1}(\Psi_{g_2} z))) \\ &= (j(g_1)^{-k} F \circ \Psi_{g_1})|_k(g_2)(z) = (F|_k(g_1))|_k(g_2(z) \end{aligned}$$

Exercise 3 (The hyperbolic metric)**.** The *hyperbolic length* of a piece-wise C^1-path $p : [0,1] \to \mathbb{H}$ is defined by

$$L(p) = \int_0^1 \frac{|p'(t)|}{\operatorname{Im}(p(t))} dt. \tag{0.1}$$

(a) Show that for all $g \in \mathrm{SL}_2(\mathbb{R})$ and all $p : [0,1] \to \mathbb{H}$ as above, we have $L(\Psi_g \circ p) = L(p)$. Here, we view g as an automorphism of $\mathbb{H}$ in the usual way (as in Exercise 1).

(b) The *hyperbolic metric* $\rho : \mathbb{H} \times \mathbb{H} \to [0, +\infty)$ is defined by

$$\rho(z_1, z_2) = \inf\left\{L(p) \,|\, p \in C^1_{\mathrm{pw}}([0,1], \mathbb{H}),\, p(0) = z_1,\, p(1) = z_2\right\}.$$

Check that it is indeed a metric and that it is $\mathrm{SL}_2(\mathbb{R})$-invariant.

(c) Show that $\rho(i, iy) = \log(y)$ for all $y \geq 1$.

(d) Show that for all $z_1, z_2 \in \mathbb{H}$ there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gz_1 = i$, $\operatorname{Re}(gz_2) = 0$ and $\operatorname{Im}(gz_2) \geq 1$.

(e) Let (z_1, z_2), $(z_1', z_2') \in \mathbb{H} \times \mathbb{H}$ be such that $\rho(z_1, z_2) = \rho(z_1', z_2')$. Show that there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $(gz_1, gz_2) = (z_1', z_2')$.

(f) Show that for all $z_1, z_2 \in \mathbb{H}$ we have

$$\cosh(\rho(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2 \operatorname{Im}(z_1) \operatorname{Im}(z_2)}.$$

Hint: Use invariance under the diagonal action of $\mathrm{SL}_2(\mathbb{R})$ of both sides to check the identity only for an easy set.

(g) Show that the topology on $\mathbb{H}$ induced by the hyperbolic metric is equal to the subspace topology (of $\mathbb{H}$ as an open subset of $\mathbb{C}$).

Solution. (a) Let $p\colon [0,1] \to \mathbb{H}$ be any piecewise C^1-path and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Then

$$\begin{aligned} \frac{|(\Psi_g \circ p)'(t)|}{\operatorname{Im}(\Psi_g \circ p)(t))} &= \frac{|\Psi_g'(p(t))||p'(t)|}{\frac{\operatorname{Im}(p(t))}{|cp(t)+d|^2}} \\ &= \frac{p'(t)}{\operatorname{Im}(p(t)}. \end{aligned}$$

Already the integrand is g-invariant and so also the integral.

(b) The map is clearly well defined. It is easy to see that it is symmetric and the triangle inequality is satisfied. The delicate bit is to show that the metric is positive definite, i.e. that if $z_1 \neq z_2 \in \mathbb{H}$, then $\rho(z_1, z_2) > 0$. This follows e.g. from points (c),(d) and g-invariance of ρ. We can also prove it directly:

Let consider $z_1 \neq z_2 \in \mathbb{H}$ and $p\colon [0,1] \to \mathbb{H}$ be a piecewise C^1-path between z_1 and z_2. Consider the euclidean open ball $B_r(z_1) = \{z \in \mathbb{H}, |z - z_1| < r\}$ with $0 < r < \frac{1}{2}\min(\mathrm{Im}(z_1), |z_2 - z_1|)$. Notice that for every $z \in B_r(z_1)$ we have

$$|\mathrm{Im}(z)| \leq \frac{3}{2}|\mathrm{Im}(z_1)|$$

Let $t_0 = \inf\{t \in [0,1], p(t) \notin B_r(z_1)\}$. By continuity of p the set is non-empty, $t_0 > 0$ and $|p(t_0) - z_1| \geq r$. In particular we have

$$\begin{aligned} L(p) &\geq \int_0^{t_0} \frac{|p'(t)|}{\mathrm{Im}(p(t)} \,\mathrm{d}t \\ &\geq \frac{2}{3\,\mathrm{Im}(z_1)} |\int_0^{t_0} p'(t) \,\mathrm{d}t| \\ &\geq \frac{2}{3\,\mathrm{Im}(z_1)} r. \end{aligned}$$

Being r and $\mathrm{Im}(z_1)$ independent of ρ we see by taking the infimum that $\rho(z_1, z_2) > 0$.

(c) Consider the path $p\colon t \in [0,1] \mapsto iyt + (1-t)i$. Then

$$\begin{aligned} L(p) &= \int_0^1 \frac{y-1}{t(y-1)+1} \,\mathrm{d}t \\ &= \int_1^y \frac{1}{u} \mathrm{d}u = \log(y). \end{aligned}$$

On the other hand for any path p connecting i and iy we have

$$\begin{aligned} L(p) &\geq \int_0^1 \frac{|\mathrm{Im}(p'(t))|}{\mathrm{Im}(p(t)} \,\mathrm{d}t \\ &\geq \log(\mathrm{Im}(p(1))) - \log(\mathrm{Im}(p(0)) = \log(y). \end{aligned}$$

(d) We first apply Exercise 1 to find a $g \in \mathrm{SL}_2(\mathbb{R})$ so that $gz_1 = i$. Hence it suffices find $h \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i)$ so that $hz_2 = iy$ for some $y \geq 1$. From exercise 1 we (basically) know that

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R} \right\} = \mathrm{SO}(2)(\mathbb{R}).$$

It is easy to compute, for $k(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, that

$$\mathrm{Re}(k(\theta) \cdot z_2) = \frac{\mathrm{Re}(z_2)(\cos^2\theta - \sin^2\theta) - \cos(\theta)\sin(\theta)|z_2|^2}{|cos\theta - \sin\theta z_2|^2}$$

We have then $\mathrm{Re}(k(0)z_2) = \mathrm{Re}(z_2)$ and $\mathrm{Re}(k(\pi/2)z_2) = -\frac{\mathrm{Re}(z_2)}{|z_2|^2}$. The map $\theta \mapsto k(\theta)z_2$ is clearly continuous and hence there exists a θ_0[1] statisfing $\mathrm{Re}(k(\theta_0)z_2)) = 0$. If $k(\theta_0)z_2 = iy$ for some $0 < y < 1$ we can apply $k(\pi/2)$ to iy to obtain iy^{-1}.

(e) Find $g, g' \in \mathrm{SL}_2(\mathbb{R})$ so that $gz_1 = i = g'z_1'$ and $gz_2 = iy, g'z_2' = iy' \in i\mathbb{R}_{>1}$. By comparing the distances we deduce $\log(y) = \log(y')$. Hence $y = y'$. In particular it follows that $g'^{-1}g \cdot (z_1, z_2) = (z_1', z_2')$

[1] This is a bit lazy, θ_0 can actually be computed.

(f) We check that the right hand side is invariant under the diagonal action of $g \in \mathrm{SL}_2(\mathbb{R})$:

$$\frac{|gz_1 - gz_2|^2}{2\,\mathrm{Im}(gz_1)\,\mathrm{Im}(gz_2)} = \frac{\frac{|(az_1+b)j(g)(z_2)-(az_2+b)j(g)(z_1)|^2}{|j(g)(z_1)j(g)(z_2)|^2}}{\frac{\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)}{|j(g)(z_1)j(g)(z_2)|^2}},$$

where as in the previous exercise $j(g)(z) = cz + d$. Moreover we have

$$(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = z_1(ad - bc) - z_2(ad - bc) = z_1 - z_2.$$

In particular it suffices to show the equality for $z_1 = i$ and $z_2 = iy$ for $y \geq 1$ and indeed:

$$\begin{aligned}\cosh(\rho(i, iy)) &= \cosh(\log(y)) \\ &= \frac{y + y^{-1}}{2} \\ &= 1 + \frac{(y-1)^2}{2y}.\end{aligned}$$

(g) Let $d : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_{\geq 0}$, $d(z, z') = |z - z'|$ denote the Euclidean metric . For all $z \in \mathbb{H}$ and $r > 0$ we define the open balls with center z and radius, with respect to d and ρ repetitively, by

$$B_r^d(z) := \{w \in \mathbb{H} : d(z, w) < r\}, \quad B_r^\rho(z) := \{w \in \mathbb{H} : \rho(z, w) < r\}$$

Fix z_0 and $r > 0$. We must show that:

(i) The set $B_r^\rho(z_0)$ is open in the Euclidean topology,

(ii) The set $B_r^d(z_0)$ is open in the topology induced by ρ.

To prove (ii), consider a point $w_0 \in B_r^\rho(z_0)$. Write $\rho(z_0, w_0) = r - \delta$ where $\delta \in (0, r]$. We want to find $\varepsilon > 0$ such that for all $z \in \mathbb{H}$:

$$d(z, w_0) < \varepsilon \quad \Rightarrow \quad \rho(z_0, z) < r.$$

(Because this means that $B_\varepsilon^d(w_0) \subset B_r^\rho(z_0)$). Since ρ satisfies the triangle inequality, it suffices to find $\varepsilon > 0$ such that for all $z \in \mathbb{H}$

$$d(z, w_0) < \varepsilon \quad \Rightarrow \quad \rho(z, w_0) < \delta.$$

The existence of such an $\varepsilon > 0$ follows from the explicit formula for ρ given in 4 f), which shows that ρ is continuous with respect to the Euclidean topology.

We turn to the proof of (i). The beginning of the argument is exactly as in part (ii), with the roles of d and ρ reversed. Consider $w_0 \in B_r^d(z_0)$. Write $d(z_0, w_0) = r - \delta$ where $\delta \in (0, r]$. We want to find $\varepsilon > 0$ such that for all $z \in \mathbb{H}$:

$$\rho(z, w_0) < \varepsilon \quad \Rightarrow \quad d(z_0, z) < r.$$

(Because this means that $B_\varepsilon^\rho(w_0) \subset B_r^d(z_0)$). Since d satisfies the triangle inequality, it suffices to find $\varepsilon > 0$ such that for all $z \in \mathbb{H}$

$$\rho(z, w_0) < \varepsilon \quad \Rightarrow \quad d(z, w_0) < \delta.$$

Again by the explicit formula 4f), if $\rho(z, w_0)$ is small, then $\frac{|z-w_0|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w_0)}$ is small, so it suffices to find $\varepsilon' > 0$, $\varepsilon'' > 0$ such that for all $z \in \mathbb{H}$

$$\left(\rho(z, w_0) < \varepsilon'' \quad \text{and} \quad \frac{|z - w_0|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w_0)} < \varepsilon'\right) \quad \Rightarrow \quad |w - z_0| < \delta. \tag{0.2}$$

We the general estimate $\rho(z, w_0) \geq |\,\mathrm{Im}(z) - \mathrm{Im}(w_0)|$ from before and deduce

$$\left(\rho(z, w_0) < \varepsilon'' \quad \text{and} \quad \frac{|z - w_0|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w_0)} < \varepsilon'\right) \quad \Rightarrow \quad \frac{|z - z_0|^2}{(\mathrm{Im}(w_0) + \varepsilon'')\,\mathrm{Im}(w_0)} \leq \quad \frac{|z - w_0|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w_0)} < \varepsilon'.$$

It is now clear that $\varepsilon', \varepsilon'' > 0$ exist such that (0.2) holds for all $z \in \mathbb{H}$. This finishes the proof.

Exercise 4 (The hyperbolic measure)**.** (a) For $z \in \mathbb{H}$ we write $z = x + iy$, $x, y \in \mathbb{R}$, $y > 0$. By quoting a result from measure and integration theory or otherwise, give a precise meaning to: *the measure* μ *on* $\mathbb{H}$ *given by* $d\mu(z) = \frac{dxdy}{y^2}$. Show that this measure μ is $\mathrm{SL}_2(\mathbb{R})$-invariant, that is, show that for all $f \in C_c(\mathbb{H})$ and all $g \in \mathrm{SL}_2(\mathbb{R})$, we have

$$\int_{\mathbb{H}} f d\mu = \int_{\mathbb{H}} (f \circ \Psi_g) d\mu,$$

where Ψ_g is defined as in Exercise 1.

(b) Compute the volume with respect to μ of the following set

$$\left\{ z \in \mathbb{H},\ |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2} \right\}.$$

Solution. (a) To give a precise definition of μ, we first recall a few definitions and general facts from measure theory (taken form [2, ch. 3.1]). Let X be a locally compact, second countable and Hausdorff topological space. Let $C_c(X) = C_c(X, \mathbb{R})$ denote the space of real-valued, compactly supported continuous functions on X. Recall that a linear form $C_c(X) \to \mathbb{R}$ is *positive*, if $\Lambda(f) \geq 0$ holds for all everywhere non-negative $f \in C_c(X)$. Let $\mathcal{B} \subset 2^X$ denote the Borel sigma algebra. Recall that a *Radon* measure on X is a measure $\mu : \mathcal{B} \to [0, +\infty]$, which is finite on all compact subsets and inner regular on all Borel sets $B \in \mathcal{B}$. The latter means that $\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}$. To each such Radon measure μ, one can attach a positive linear form Λ_μ, defined by $\Lambda_\mu(f) = \int_X f d\mu$ for $f \in C_c(X)$. The *Riesz representation theorem* (see [2][Theorem 3.15]) asserts that the assignment $\mu \mapsto \Lambda_\mu$ defines a bijection between

- the set of all Radon measures on X,
- the set of all positive linear forms on $C_c(X)$.

Thus, one can (implicitly) define a measure on X by defining a positive linear form on $C_c(X)$.

We apply the above discussion to $X = \mathbb{H}$. For this exercise, we denote by $\lambda : X \to [0, +\infty]$ the restriction of the Lebesgue measure on $\mathbb{R}^2$. For $f \in C_c(X)$, we define $\widehat{f} \in C_c(X)$ by[2]

$$\hat{f}(z) := \frac{f(z)}{\mathrm{Im}(z)^2}.$$

We define

$$\Lambda : C_c(X) \to \mathbb{R}, \qquad \text{by} \qquad \Lambda(f) := \int_X \widehat{f} d\lambda.$$

It is clear that Λ is a well-defined, positive linear form on $C_c(X)$.

Definition 1. The *hyperbolic measure* μ is the unique Radon measure on $X = \mathbb{H}$ corresponding to the linear form Λ defined as above, via the Riesz representation theorem.

Now we prove the invariance. Let $g \in \mathrm{SL}_2(\mathbb{R})$. Let Ψ_g be as in Exercise 2. It defines a C^∞-diffeomorphism on $\mathbb{H}$. Write $D\Psi_g(z) : \mathbb{C} \to \mathbb{C}$ for the derivative of Ψ_g at the point $z \in \mathbb{H}$. This is an invertible $\mathbb{R}$-linear map. Its determinant is

$$\det\left(D\Psi_g(z)\right) = |(\Psi_g)'(z)|^2, \tag{0.3}$$

where, on the right, the derivative $(\Psi_g)'(z) = \lim_{\mathbb{C}^\times \ni h \to 0} \frac{\Psi_g(z+h) - \Psi_g(z)}{h} \in \mathbb{C}^\times$ is the complex derivative. We compute that

$$(\Psi_g)'(z) = \frac{1}{(cz+d)^2} \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{0.4}$$

[2]There's no relation to the Fourier transform here, but the notation seemed typographically convenient.

We claim that

$$\widehat{f \circ \Psi_g}(z) = \widehat{f}(\Psi_g(z))|\det(D\Psi_g(z))| \quad \text{for all } z \in \mathbb{H}. \tag{0.5}$$

Granting (0.5) and transformation formula for integrals (change of variables theorem, see [2, Thm 2.16]) we deduce that, for every $f \in C_c(X)$,

$$\begin{aligned}
\int_X (f \circ \Psi_g) d\mu &= \Lambda(f \circ \Psi_g) && \text{by definition of } \mu \\
&= \int_X \widehat{f \circ \Psi_g}(z) d\lambda(z) && \text{by definition of } \Lambda \\
&= \int_X \widehat{f}(\Psi_g(z))|\det(D\Psi_g(z))| d\lambda(z) && \text{by (0.5)} \\
&= \int_X \widehat{f}(w) d\lambda(w) && \text{by the transformation theorem and } \Psi_g(X) = X \\
&= \Lambda(f) && \text{by definition of } \Lambda \\
&= \int_X f d\mu && \text{by definition of } \mu,
\end{aligned}$$

which is the desired.

(b) We have

$$\{z \in \mathbb{H}, |z| \geq 1, |\mathrm{Re}(z)| \leq 1/2\} = \{z \in \mathbb{H}, \mathrm{Im}(z) \geq \sqrt{1 - \mathrm{Re}(z)^2}, -1/2 \leq \mathrm{Re}(z) \leq 1/2\}.$$

$$\int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \, \mathrm{d}y \, \mathrm{d}x = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \frac{\pi}{3}$$

References

[1] Fred Diamond, Jerry Shurman, *A first course in modular forms.* Springer.

[2] Dietmar Salamon, *Measure and Integration*, European Mathematical Society, EMS Textbooks in Mathematics.

[3] M. Einsiedler and T. Ward *Ergodic Theory with a view towards Number Theory*, Springer Graduate Texts in Mathematics, 2010