Proposition: Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is an isomorphism if and only if so is φ_P for every P.

Proof: $\Rightarrow$) immediate: since φ is an iso we have its inverse φ^{-1}, and one checks that $(\varphi^{-1})_P = (\varphi_P)^{-1}$ for every P.

$\Leftarrow$) For any U, $\varphi(U)$ is a homomorphism. So to have an inverse homomorphism $(\varphi(U))^{-1}$, it suffices that $\varphi(U)$ is bijective (the set theoretic inverse will automatically be an homomorphism).

Then, since for every $V \subseteq U$, we have

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

commutes

once $(\varphi(U))^{-1}$ and $(\varphi(V))^{-1}$ exist, it follows immediately that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[l, "(\varphi(U))^{-1}"'] \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) & \mathcal{G}(V) \arrow[l, "(\varphi(V))^{-1}"] \end{tikzcd}$$

commutes

So, to conclude, we only need to show $\varphi(U)$ is bijective for every U.

① φ_p injective for all $P \Longrightarrow \varphi(U)$ inj for all U

Fix U open and $s \in \mathcal{F}(U)$ s.t. $\varphi(U)(s) = 0 \in \mathcal{G}(U)$.
Then, for any $P \in U$

$$\mathcal{G}_p \ni 0_p = \big(\varphi(U)(s)\big)_p \underset{\substack{\uparrow \\ \text{Remark}}}{=} \varphi_p(s_p)$$

Since φ_p is injective and $\varphi_p(s_p) = 0_p$, we have $s_p = 0_p \in \mathcal{F}_p$.
So, for all $P \in U$, $s_p = 0$. By the Lemma, $s = 0$

② φ_p surjective for all P & $\varphi(U)$ injective for all U $\Longrightarrow$ $\varphi(U)$ surjective for all U

Fix U open and $t \in \mathcal{G}(U)$.
Fix $P \in U$. Since φ_p is surjective, there is a germ $\sigma_p \in \mathcal{F}_p$ s.t. $\varphi_p(\sigma_p) = t_p$.
By definition of stalk, we may find an open U_p with $P \in U_p \subseteq U$ and a section $s^{(p)} \in \mathcal{F}(U_p)$ s.t. $(s^{(p)})_p = \sigma_p$.
Then, $\varphi(U_p)(s^{(p)}) \in \mathcal{G}(U_p)$, $t|_{U_p} \in \mathcal{G}(U_p)$, and
$(t|_{U_p})_p = t_p = \varphi_p(\sigma_p) = \big(\varphi(U_p)(s^{(p)})\big)_p$

piece missing in class to go from = at stalk to = on an open

Since $t|_{U_p}$ and $\varphi(U_p)(s^{(p)})$ have the same germ at P, there is an open V_p with $P \in V_p \subseteq U_p$ s.t.

$$t|_{V_p} \underset{\substack{\uparrow \\ \text{axioms of sheaf}}}{=} (t|_{U_p})|_{V_p} = \big(\varphi(U_p)(s^{(p)})\big)\big|_{V_p} \underset{\substack{\uparrow \\ \text{morphism}}}{=} \varphi(V_p)(s^{(p)}|_{V_p})$$

So, up to shrinking U_P to V_P and replacing $s^{(P)}$ with $s^{(P)}|_{V_P}$, we may assume

$$\varphi(U_P)(s^{(P)}) = t|_{U_P}$$

Comment: that is, mathematically, we ~~also~~ need to work with V_P and $s^{(P)}|_{V_P}$. Yet, V_P and $s^{(P)}|_{V_P}$ have all the properties that we wanted from U_P and $s^{(P)}$ plus an extra property. So, to keep the notation light, we replace.

Now, $U = \bigcup_{P \in U} U_P$.

Fix $P, P' \in U$. Then, we have

$$t|_{U_P \cap U_{P'}} = \left(\varphi(U_P)(s^{(P)})\right)\Big|_{U_P \cap U_{P'}} = \varphi(U_P \cap U_{P'})\left(s^{(P)}|_{U_P \cap U_{P'}}\right)$$

$$= \left(\varphi(U_{P'})(s^{(P')})\right)\Big|_{U_P \cap U_{P'}} = \varphi(U_P \cap U_{P'})\left(s^{(P')}|_{U_P \cap U_{P'}}\right)$$

Since $\varphi(U_P \cap U_{P'})$ is injective, we have $s^{(P)}|_{U_P \cap U_{P'}} = s^{(P')}|_{U_P \cap U_{P'}}$

By the gluing axiom, there is $s \in \mathcal{F}(U)$ such that, for every P, $s|_{U_P} = s^{(P)}$. In particular, for every P, $s_P = \sigma_P$.

Then, for every P we have

$$t_P = \varphi_P(\sigma_P) = \varphi_P(s_P) = \left(\varphi(U)(s)\right)_P$$

By the lemma, we have $t = \varphi(U)(s)$. $\blacksquare$