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AG II - Schemes and sheaves

Solutions – week 8

Exercise 1. *Closed subschemes.*

(1) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf. Let $Z = \mathrm{supp}(\mathcal{O}_X/\mathcal{I})$. Denote by $\iota : Z \to X$ the inclusion. Show that $(Z, \iota^*\mathcal{O}_X/\mathcal{I})$ is a scheme. In what follows, $V(\mathcal{I})$ denotes the above associated scheme.
Hint: This is a local question. So using that $\mathcal{I}$ is quasi-coherent you can reduce to the case where X is affine and $\mathcal{I}$ correspond to an ideal.

(2) Show that $V(\mathcal{I})$ is a closed subscheme of X.

(3) Show that

$$\{\text{Quasi-coherent ideals of } \mathcal{O}_X\} \longleftrightarrow \{\text{Closed subschemes of } X\}$$

sending $\mathcal{I}$ to $V(\mathcal{I})$ is a one-to-one correspondence.

(4) Let $\mathrm{Spec}(A)$ be an affine scheme. Show that

$$\{\text{Ideals of } A\} \longleftrightarrow \{\text{Closed subschemes of } \mathrm{Spec}(A)\}$$

sending $I \mapsto (\mathrm{Spec}(A/I) \to \mathrm{Spec}(A))$ is a one-to-one correspondence.
Hint: You may use the equivalence of categories between quasi-coherent sheaves of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules and A-modules. For a proof which does not use this fact, see the solution of exercise 5, week 6.

Exercise 2. *Intersection of affine schemes.* Let X be a scheme and $U, V \subset X$ be open affine sub-schemes.

(1) Show that if X is separated then $U \cap V$ is affine.
Hint: Show that $U \cap V \cong X \times_{X\times X} (U \times V)$.

(2) Show that $U \cap V$ is not necessarily affine if X is not separated.
Hint: remember this open of an affine which is not affine? Play with this.

Solution key. For the first point, the claim follows from the Hint because the intersection is realized has a closed subscheme of an affine scheme. For the second point, one can take the affine plane with two origins. □

Exercise 3. *A map from a proper scheme to a separated scheme is closed.* Let $f\colon X \to Y$ be a map of S-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph $(\mathrm{id}, f) = \Gamma_f\colon X \to X \times_S Y$ is a closed immersion.

(2) Let $Z \subset X$ a closed subscheme proper over S. Show that $f_{|Z}$ is closed.

Solution key. The first point follows because

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \Gamma_f}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_S Y & \xrightarrow{f \times \mathrm{id}} & Y \times_S Y \end{array}$$

is a pullback square. The second claim follows because $Z \to Z \times_S Y \to Y$ is closed, the first map being closed by the first point and the second map being closed by universal closedness of $Z \to S$.

□

Remark. This fact is analogue to the topological result that a continuous map from a compact topological space to a Hausdorff space is always closed.

Exercise 4. *Morphisms into separated schemes.* Let S be a scheme. Let $X \to S$ and $Y \to S$ be S-schemes. Suppose that X is reduced and $Y \to S$ separated. Show that two morphisms of S-schemes

$$f_1, f_2 \colon X \to Y$$

that coincide an open dense subset of X are equal.
Give counter-examples if one of the hypotheses is dropped.

Solution key. Let Z be the scheme where $f_1 = f_2$ *i.e.* the pullback

$$\begin{array}{ccc} Z & \xrightarrow{f_1 = f_2} & Y \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{f_1 \times f_2} & Y \times_S Y \end{array}$$

Because Y is separated Z is closed in X. Because of the assumption, $Z = X$ topologically, but then schematically because X is reduced.
We provide a counter-example if X is not reduced. Consider the two k-algebras maps (k being a field say)

$$k[x] \mapsto k[x, y]/(xy, y^2)$$

sending x to x and $x + y$ respectively. The induced maps on Spec agree on $D(x)$ which is dense. □

Remark. This fact is analogue to the topological result that if two continuous morphisms to a Hausdorff space agree on a open dense then they actually agree everywhere.

Exercise 5. *Generically finite morphisms.*

(1) Let k be a field. If $k \to A$ is finite, show that every prime of A is maximal.

Let $f \colon X \to Y$ be a dominant morphism between integral schemes.

(2) If f is finite, show that $\dim(X) = \dim(Y)$.
Hint: reduce to the affine case. Then use going up and that the map is surjective. Use point (1) to deduce that if $A \to B$ is finite and the preimage of two primes in B is the same in A, then the two primes are not included in one another.

(3) If f is finite type and $K(Y) \subset K(X)$ is a finite extension of fields, show that there exists an non-empty open $U \subset Y$ such that $f\colon f^{-1}(U) \to U$ is a finite morphism.
Hint: first prove the case where both X and Y are affine and then battle to use this case to conclude.

Solution key. (1) Because A is a finite dimensional k-vector space, it is Noetherian and Artinian. Let J be the Jacobson radical of A. Because of the Artinian hypothesis, $J^n = J^{n+1}$ for some n. Then by Nakayama $J^n = 0$. It implies that the Jacobson radical is equal to the radical.

Also, A has a finite number of maximal ideals. Indeed if not say $(\mathfrak{m}_i)$ is an infinite list of distinct maximal ideals. Then by the Artinian property, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{r+1}$. But then $\mathfrak{m}_1 \cdots \mathfrak{m}_r \subset \mathfrak{m}_{r+1}$, a contradiction (see a similar argument below). Say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals.

If $\mathfrak{p}$ is prime, then $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r \subset J \subset \mathfrak{p}$. If for all $\mathfrak{m}_i$ we have $\mathfrak{m}_i \not\subset \mathfrak{p}$, then we have elements $x_i \in \mathfrak{m}_i \notin \mathfrak{p}$. But then $x_1 \cdots x_r \in \mathfrak{p}$ a contradiction.

(2) The dimension is equal to the one of a dense open. So we can reduce to the affine case. Now, let $A \to B$ be finite between integral domains. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ a maximal chain of primes in A, meaning that there is no prime lying in between those. Because the map is finite, the map on Spec is surjective so we have $\mathfrak{q}_0$ lying over $\mathfrak{p}_0$. By going up we can lift to a chain in B. We now argue that we can not fit any more primes in the above list. If so, we would have two primes $\mathfrak{q} \subset \mathfrak{q}'$ with the same image in $\mathrm{Spec}(A)$. But by the previous point if two primes are in the same fiber, because the fiber has only maximal ideals, we see that $\mathfrak{q} = \mathfrak{q}'$.

(3) Note that without loss of generality Y is affine. We first begin by supposing that X is also affine. We are then in the situation of a finite type map $A \to B$ between integral domains, such that this map induces a finite extension of fields at the fields of fractions. Say that $b_1, \ldots, b_n$ generates B as an A-algebra. By hypothesis, there exists polynomials $f_i \in \mathrm{Frac}(A)[t]$ such that $f_i(b_i) = 0$. Therefore there exist a non-zero element $g \in A$, namely the product of the denominators of each coefficients of the polynomials f_i, such that $b_1, \ldots, b_n$ are integral over A_g. This implies that $A_g \to B_g$ is finite.

Now we show the general case. As $f\colon X \to Y$ is finite type over an affine scheme, there exists a finite covering by affine schemes X_i of X. By the preceding case, there exists $U_i \subset Y$ such that $f^{-1}(U_i) \cap X_i \to U_i$ is finite. We may replace Y by the intersection of the U_i's and also suppose that it is affine. With this reduction, we are now in the following situation: we have a covering X_i of X

such that $X_i \to Y$ is finite. Let V be the intersection of the X_i's. Say that $A \to B$ is the finite ring map corresponding to $X_1 \to Y$. Say that $0 \neq b \in B$ is such that $D(b) \subset V$. Note that as b is integral over A, there is a polynomnial with non zero-constant coefficient $\sum_{i=0}^{n} a_i t^i \in A[t]$ such that $f(b) = 0$. Therefore,

$$b\left(\sum_{i=1}^{n} a_i b^{i-1}\right) = -a_0$$

Therefore, there is a non-zero element $b' \in B$ such that $a := bb' \in A$. It implies that $f^{-1}(D(a)) \subset D(b) \subset V \subset X_1$. Therefore $f\colon f^{-1}(D(a)) \to D(a)$ is finite.

□

Exercises 6, 7, and 8 are purely about the underlying topology of the schemes in question.

Exercise 6. *Projection from affine spaces.* Let R be a ring.

(1) Show that

$$\pi\colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$

is open. More precisely, if $f = \sum a_i t^i$ show that

$$\pi\left(D(f(t))\right) = \bigcup_i D(a_i).$$

(2) Let $g(t) \in R[t]$ be a monic polynomial and $f(t) \in R[t]$. Remark that $R[t]/g(t)$ is a free R-module of rank $\deg(g)$. Let $\chi(X) = \sum_i^n r_i X^i$ be the characteristic polynomial of the multiplication by $f(t)$ on $R[t]/g(t)$. Show that

$$\pi(D(f) \cap V(g)) = \bigcup_i^{n-1} D(r_i).$$

Solution key. (1) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\mathfrak{p} \in \pi(D(f(t)))$ if and only if $k(\mathfrak{p})[t]_{f(t)} \neq 0$ if and only if $f(t) \neq 0$ in $k(\mathfrak{p})[t]$ if and only if there is some i such that $a_i \notin \mathfrak{p}$.

(2) Let $\mathfrak{p} = \mathfrak{q} \cap R$ with $\mathfrak{q} \in D(f) \cap V(g)$ in the image. So we have a map $k(\mathfrak{p})[t]/(g(t)) \to k(\mathfrak{q})$. Note the following fact: by Cayley-Hamilton f is nilpotent in $k(\mathfrak{p})[t]/(g(t))$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.

Note also that $f \neq 0$ in $k(\mathfrak{q})$ because $\mathfrak{q} \in D(f)$. So f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$ and therefore $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$.

Reciprocally if $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$, the by the above argument f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$. Therefore there is some $\mathfrak{q} \notin f$ in $k(\mathfrak{p})[t]/(g(t))$ meaning that $\mathfrak{q} \in D(f) \cap V(g)$, which is therefore sent to $\mathfrak{p}$.

□

Exercise 7. *Chevalley's theorem.* Let X be a Noetherian topological space. A subset $T \subset X$ is called *constructible* if it can be written as a finite union of sets of the form $U \cap V^c$ where U and V are open sets.

(1) Show that if $X = \mathrm{Spec}(R)$ for a Noetherian ring R, a subset is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ with $f, g_1, \ldots, g_m \in R$.

(2) Show using exercise 6 that

$$\pi\colon \mathrm{Spec}(R[t]) \to \mathrm{Spec}(R)$$

sends constructible subsets to construtible subsets.
Hint: Show by induction on $\sum_i \deg(g_i)$ that if $f, g_1, \ldots, g_m \in R[t]$ are polynomials, the image of $D(f) \cap V(g_1, \ldots, g_m)$ is constructible. To conduct the induction step, consider α the leading coefficient of g_1. Break down the study on the open and closed $D(\alpha)$ and $V(\alpha)$ to reduce the sum of the degrees.

(3) Deduce *Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type morphism between Noetherian schemes. Then f sends constructible subsets to constructible subsets.[1]

Solution key. (2) Note that we already know two cases. Namely the case $D(f)$ and the case $D(f) \cap V(g)$ where g is monic. We proceed by induction on the sum of the degrees of g_i – also we order them such that they have increasing degrees. Let c be the dominant coefficient of g_1. We have

$$\mathrm{Spec}(R[t]) = \mathrm{Spec}(R/c[t]) \sqcup \mathrm{Spec}(R_c[t]).$$

In the first the image of g is of degree striclty less. So induction goes.

Also, note that g_1 is monic in $\mathrm{Spec}(R_c[t])$. If $n = 1$, we are in an already dealt situation. If not let

$$g_2' = g_2 - t^{d_2 - d_1}(c'/c)g_1$$

where c' is the leading coefficient of g_1. Then

$$D(f) \cap V(g_1, g_2, \ldots, g_n) = D(f) \cap V(g_1, g_2', \ldots, g_n).$$

But now the sum of degrees of the list lowers giving the claim by induction.

(3) We can reduce to the affine case where we can reduce to

$$R \to R[t_1, \ldots, t_n] \to S$$

where the last map is surjective. The first arow induces on Spec a map which preserves constructibility by the above and the second also because it is a closed immersion

□

Remark. In general the topological image of a morphism of schemes can fail to be open or closed but in cases where Chevalley's theorem applies, it

[1]The generalization to non-Noetherian settings requires more careful definitions, but once these definitions are addressed the proof is the same.

tells that it still not too far from it and manageable. In particular one can endow the image with a scheme structure.

Exercise 8. *An application of Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type dominant map between Noetherian schemes with Y irreducible. Use Chevalley's theorem to show that the topological image $f(X)$ contains an open set.

Solution key. The image being dense contains the generic point of Y, therefore, $\eta_Y \in U \cap Z \subset f(X)$ because the topological image $f(X)$ is constructible, for some open U and closed Z of Y. But if $\eta_Y \in Z$ then we see that $Z = Y$ $\square$