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AG II - Schemes and sheaves

Solutions – week 14

Exercise 1. *A short exact sequence.* Let $\iota\colon D \to X$ be an effective Cartier divisor on a integral scheme X. Show that there is a short exact of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \iota_* \mathcal{O}_D \to 0.$$

Deduce that there is a long exact sequence in cohomology,

$$(\dots) \to \mathrm{H}^i(X, \mathcal{O}(-D)) \to \mathrm{H}^i(X, \mathcal{O}_X) \to \mathrm{H}^i(D, \mathcal{O}_D) \to (\dots)$$

Solution key. $\mathcal{O}(-D)$ is the ideal sheaf associated to the closed subscheme D.

□

Exercise 2. Let $\iota : Z \to X$ be a closed immersion of schemes, where Z and X are not necessarily noetherian schemes.

(1) Show that the functors $R^j \iota_* : \mathrm{Mod}_{\mathcal{O}_Z} \to \mathrm{Mod}_{\mathcal{O}_X}$ are zero for all $j > 0$.
(2) Conclude that for an $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_Z}$, $\mathrm{H}^i(Z, \mathcal{F}) \cong \mathrm{H}^i(X, \iota_* \mathcal{F})$ for all $i \in \mathbb{N}$.

Solution key. The functor j_* is exact, which can be checked at stalks. The assertion on higher pushforwards follows.
Consider morphisms of ringed spaces (the right one is the terminal ringed space)

$$(Z, \mathcal{O}_Z) \xrightarrow{\iota} (X, \mathcal{O}_X) \xrightarrow{p} (*, \mathbb{Z}).$$

It holds in general that $R(p \circ \iota)_* = Rp_* \circ R\iota_*$ at the level of derived categories of $\mathcal{O}$-modules.
As the first point shows that $R\iota_* = \iota_*$, this concludes.

□

Exercise 3. *A geometric perspective on the Euler sequence.* Let A be a ring and M a locally free of finite rank A-module.

(1) *Directional derivative.* For a $v \in M$, show that there is a unique A-derivation

$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee) \to \mathrm{Sym}(M^\vee)$$

which is equal to the evaluation at v on elements of degree 1. If M is free, if (e_i) and (x_i) denotes a basis and a dual basis respectively, and $v = \sum \lambda_i e_i$, show that

$$\frac{\partial}{\partial v} = \sum_i \lambda_i \frac{\partial}{\partial x_i}.$$

(2) For $\varphi \in M^\vee$, show that $\frac{\partial}{\partial v}$ uniquely extends to an A-derivation

$$\frac{\partial}{\partial v}\colon \operatorname{Sym}(M^\vee)_\varphi \to \operatorname{Sym}(M^\vee)_\varphi.$$

Deduce that $\frac{\partial}{\partial v}$ defines an A-derivation

$$\frac{\partial}{\partial v}\colon \operatorname{Sym}(M^\vee)_{(\varphi)} \to \operatorname{Sym}(M^\vee)(-1)_{(\varphi)}.$$

(3) Denote by $\pi\colon \mathbb{P}(M) \to \operatorname{Spec}(A)$ and $\mathcal{T}^1_{\mathbb{P}(M)|A} = \left(\Omega^1_{\mathbb{P}(M)|A}\right)^\vee$. Deduce that there is a $\mathcal{O}_{\mathbb{P}(M)}$-linear map

$$\frac{\partial}{\partial(-)}\colon \pi^* M \to \mathcal{T}^1_{\mathbb{P}(M)|A}(-1).$$

Hint: $\mathcal{T}^1_{\mathbb{P}(M)|A}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(M)}}(\Omega^1_{\mathbb{P}(M)|A}, \mathcal{O}(-1))$[1]. *Use the universal property of* $\Omega^1_{\mathbb{P}(M)|A}$ *on affines* $D_+(\varphi)$.

(4) *Euler sequence.* Let S be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank on S. Show that there is an exact sequence of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-locally free sheaves

$$0 \to \mathcal{O}(-1) \to \pi^*\mathcal{E} \xrightarrow{\frac{\partial}{\partial(-)}} \mathcal{T}^1_{\mathbb{P}(\mathcal{E})|S}(-1) \to 0$$

where the first arrow is the canonical inclusion $\mathcal{O}(-1) \to \pi^*\mathcal{E}$ and the second is a globalization of the arrow above. *Hint: Use the naturality of the construction to reduce to a case where the base is affine and* $\mathcal{E}$ *is free. We are now working in* $\mathbb{P}^n_A$. *Choose a basis and write the matrices of maps in question on opens* $D_+(x_i)$.

Solution key. (1) Because of the Leibniz rule, it suffice to determine an A-derivation on degree 1 elements.

(2) Note that because of the Leibniz rule, the following is forced.

$$0 = \varphi \frac{\partial}{\partial v}\frac{1}{\varphi} + \frac{1}{\varphi}\frac{\partial}{\partial v}(\varphi).$$

Therefore, we define

$$\frac{\partial}{\partial v}\frac{1}{\varphi} = -\frac{1}{\varphi^2}\frac{\partial}{\partial v}(\varphi) = -\frac{1}{\varphi^2}\varphi(v).$$

By the Leibniz rule, it extends. The second claim follows because such a derivation decreases the degree by 1.

(3) On an affine $D_+(\varphi)$ we define

$$M \otimes \operatorname{Sym}(M^\vee)_{(\varphi)} \to \operatorname{Der}_A(\operatorname{Sym}(M^\vee)_{(\varphi)}, \operatorname{Sym}(M^\vee)_{(\varphi)}(-1))$$

by sending $v \otimes \frac{f}{\varphi^{\deg(f)}}$ to the derivation $\frac{f}{\varphi^{\deg(f)}}\frac{\partial}{\partial v}$. This glues because this only depends on where to send M.

[1]Because in general if $\mathcal{F}$ is finite locally free and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$

(4) Working locally is enough. Indeed the sequence is functorial in $\mathcal{E}$, so this is sufficient. Let's say that $\mathcal{E} = A^{n+1}$. Fix an $i \in \{0, \dots, n\}$, without loss of generality say $i = 0$. Denote by $t_j = \frac{x_j}{x_0}$. Note that (we just need to choose where to send each t_j)

$$\mathrm{Der}_A(A[t_1, \dots, t_n], \frac{1}{x_0} A[t_1, \dots, t_n]) = \bigoplus_{j \geq 1} A[t_1, \dots, t_n] \frac{1}{x_0}.$$

In term of the above basis, we have $\frac{\partial}{\partial x_0} = (-\frac{x_1}{x_0}, \dots, -\frac{x_n}{x_0})$, because

$$\frac{\partial}{\partial x_0}(\frac{x_j}{x_0}) = \frac{-x_j}{x_0^2} = \frac{1}{x_0}\frac{-x_j}{x_0}.$$

Also we have $\frac{\partial}{\partial x_j} = e_j$. As $e_i \in (A[t_0, \dots, t_n])^{\oplus n+1}$ is sent to $\frac{\partial}{\partial x_i}$ we get that the martix of the map is

$$\begin{pmatrix} -\frac{x_1}{x_0} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_n}{x_0} & 0 & \cdots & 1 \end{pmatrix}.$$

It follows that the map is surjective and that the kernel is generated by

$$\sum_{j=0}^{n} e_j \otimes \frac{x_j}{x_0} = \frac{c}{x_0}$$

which proves the claim. (We used the notation c from the exercise on the tautological line bundle).

□

Exercise 4. *Cohomology and affine maps.* Let $X \to Y$ be an affine map of schemes and $\mathcal{F}$ a quasi-coherent sheaf on X.

(1) Show that the natural map $f_* \to Rf_*$ is an isomorphism, meaning that $R^i f_* = 0$ if $i > 0$.
(2) Deduce that
$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(Y, f_*\mathcal{F}).$$
(3) Let E be again the curve from the above exercise. Let $f\colon E \to \mathbb{P}^1_{\mathbb{C}}$ be the restriction of the partially defined projection $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_C$ on the first two components. Show that this is well defined and compute the cohomology of $f_*\mathcal{O}(nP_0)$ for $n \in \mathbb{Z}$.

Solution key. Recall that $R^i f_*$ is the sheafification of $U \mapsto \mathrm{H}^i(f^{-1}(U), \mathcal{F})$. Because f is affine, the latter vanishes, the first claim therefore follows.

Consider morphisms of ringed spaces (the right one is the terminal ringed space)

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (*, \mathbb{Z}).$$

It holds in general that $R(g \circ f)_* = Rg_* \circ Rf_*$ at the level of derived categories of $\mathcal{O}$-modules.

The first point of the exercise shows that when $\mathcal{F}$ is quasi-coherent, then $Rf_*(\mathcal{F}) = f_*\mathcal{F}$. Using this and the above we get $R(g\circ f)_*\mathcal{F} = Rg_* \circ Rf_*\mathcal{F} = Rg_*(f_*\mathcal{F})$, which concludes.
If we work on Noetherian schemes, we can use that any injective quasi-coherent is flasque. By the previous point f_* sends an injective quasi-coherent resolution of $\mathcal{F}$, to a flasque *resolution* (by exactness on quasi-coherent sheaves) of $f_*\mathcal{F}$. The second claim also follows this way in this case.

□

Exercise 5. *Curves in* $\mathbb{P}^2_k$. Let k be a field. Let $C = V_+(F)$ for $F \in \mathcal{O}_{\mathbb{P}^2_k}(d)(\mathbb{P}^2_k)$ for a $d \geq 1$.

(1) Show that $\mathrm{H}^0(C, \mathcal{O}_C) \cong k$.
(2) Deduce that any C_1 and C_2 of the above form intersect.
(3) Suppose that C does not contain $[0 : 0 : 1]$ (this can always be arranged up to an automorphism of $\mathbb{P}^2_k$). Calculate the Čech complex associated to the cover $C \cap D_+(Y) \cup C \cap D_+(X)$ explicitly and deduce that $\mathrm{H}^1(C, \mathcal{O}_C)$ is a k-vector space of dimension $\frac{(d-1)(d-2)}{2}$.

Solution key. For the first item, use that $\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}(-d)) = 0$. For the second item, suppose by contradiction that two curves $V_+(F)$ and $V_+(G)$ do not intersect. Then the union is not connected. But the union is described as $V_+(FG)$. Being disconnected, they would be non trivial idempotents in the global sections, a contradiction. We could conduct a Čech cohomology computation to get the last result, but we indicate that it follows from the long exact sequence in cohomology of

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0.$$

□

Remark. We say that $\frac{(d-1)(d-2)}{2}$ is the *arithmetic genus* of C. Curves of degree 3 are of arithmetic genus 1. Smooth ones are called *elliptic curves.* Any smooth curve C over an algebraically closed field k with $\mathrm{H}^1(E, \mathcal{O}_E) = 1$ can be realized as a smooth cubic in $\mathbb{P}^2_k$, see for example Harthshorne III,4.6.

Exercise 6. *A Čech cohomology computation.* Let k be a field. Let $U = \mathbb{A}^2_k \setminus 0$. Compute the cohomology of $\mathcal{O}_U$ on U. After showing that $\mathcal{O}_U$ is ample, deduce that Serre vanishing does not hold for U.

Solution key. The Cech complex of $\mathcal{O}_U$ with respect to the open cover $\mathcal{U} = \{D(x), D(y)\}$ is

$$\begin{array}{ccccccc}
C^0(\mathcal{O}_U) & \xrightarrow{\quad d \quad} & C^1(\mathcal{O}_U) & \longrightarrow & \mathcal{C}^2(\mathcal{O}_U) & \longrightarrow & \dots \\
\| & & \| & & \| & & \\
\mathcal{O}_U(D(x)) \times \mathcal{O}_U(D(y)) & & \mathcal{O}_U(D(xy)) & & 0 & & \\
\| & & \| & & & & \\
k[x, x^{-1}, y] \times k[x, y, y^{-1}] & & k[x, x^{-1}, y, y^{-1}] & & & & \\
& & & & & & \\
(s, t) & \longmapsto & s - t & & & &
\end{array}$$

Thus we have

$$\check{H}^1(\mathcal{U}, \mathcal{O}_U) = \frac{k[x, x^{-1}, y, y^{-1}]}{\mathrm{Im}(d)} = \frac{k[x, x^{-1}, y, y^{-1}]}{k[x, x^{-1}, y] + k[x, y, y^{-1}]} = \bigoplus_{i,j<0} \langle x^i y^j \rangle.$$

Note that $\mathcal{O}_U$ is k-very-ample. Indeed viewing U in $\mathbb{P}^2_k$, the pullback of $\mathcal{O}(1)$ on U is trivial. It now follows that $\mathcal{O}_U$ is k-very ample.

□

Exercise 7. *Coherence of derived pushforward:* Let $f : X \to Y$ be a projective morphism between two noetherian schemes. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, show that $R^i f_* \mathcal{F}$ is coherent for all i.

Solution key. Recall that if $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an inclusion of an open, then j^* sends injective $\mathcal{O}_X$-modules to $\mathcal{O}_U$-modules because it admits $j_!$ is an exact left adjoint. Therefore if $U = \mathrm{Spec}(A) \subset Y$ is open, if we consider the pullback

$$\begin{array}{ccc}
U' & \xrightarrow{j'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
U & \xrightarrow{j} & Y
\end{array}$$

then we have

$$(R^i f_* \mathcal{F})_{|U} = (R^i f'_* \mathcal{F}_{|U'}).$$

So the conclusion follows from the affine case which was seen in class as a consequence of the computation of the cohomology of line bundles $\mathcal{O}(n)$ on $\mathbb{P}^n_A$ and of the theory of ample invertible sheaves.

□