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AG II - Schemes and sheaves

Exercise to hand in. *Twistor* $\mathbb{P}^1$. (Due Sunday November 3, 18:00) Please write your solution in TeX.

Consider the following graded $\mathbb{R}$-algebra map $\sigma\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y]$

$$f(x,y) \mapsto \overline{f}(-y,x),$$

where $\overline{(-)}$ means applying the complex conjugation to the coefficients of the polynomial. Denote by $\tau = \mathrm{Proj}(\sigma)$ the induced $\mathbb{R}$-scheme morphism of $\mathbb{P}^1_{\mathbb{C}}$.

(1) Recall that we can identify the $\mathbb{C}$-rational points $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ as equivalences classes $[\alpha, \beta]$ of elements of $\mathbb{C}^2 \setminus 0$. We can also identify those with $\mathbb{C} \cup \infty$ with $[1,0] \mapsto \infty$ and $[z,1] \mapsto z$.

Describe τ on $\mathbb{C}$-rational points

$$\tau\colon \mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$$

with these two identifications of $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$.

(2) Show that $\tau^2 = \mathrm{id}$ schematically.

We now consider

$$\mathbb{C}[x,y]^\sigma = \{f(x,y) \in \mathbb{C}[x,y] \mid f(x,y) = \sigma(f(x,y)) = \overline{f}(-y,x))\}$$

the graded $\mathbb{R}$-algebra of σ-fixed points. We define the $\mathbb{R}$-scheme

$$\mathbb{P}^1_{\mathrm{tw}} = \mathrm{Proj}(\mathbb{C}[x,y]^\sigma)$$

and call it the *twistor* $\mathbb{P}^1$.

(3) Show that

$$\mathbb{C}[x,y]^\sigma = \mathbb{R}[ixy, x^2+y^2, i(x^2-y^2)].$$

(4) Show that $\mathbb{P}^1_{\mathrm{tw}} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong \mathbb{P}^1_{\mathbb{C}}$.

(5) Identify the kernel of the 2-homogeneous map of $\mathbb{R}$-algebras

$$\mathbb{R}[t_1,t_2,t_3] \to \mathbb{C}[x,y]^\sigma$$

sending $t_1 \mapsto 2ixy$, $t_2 \mapsto x^2+y^2$, $t_3 \mapsto i(x^2-y^2)$.

(6) Show that there is no $\mathbb{R}$-rational points of the twistor $\mathbb{P}^1$, *i.e.*

$$\mathbb{P}^1_{\mathrm{tw}}(\mathbb{R}) = \emptyset.$$

Remark. Some explanations and motivation.

(1) The map $\tau\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ from the first part of the exercise is what we call a *descent data*. Like saying that a sheaf can be described as a collection of sheaves on opens sets which forms a covering with isomorphisms on intersections satisfying the cocycle condition, here we specify an $\mathbb{R}$ scheme as a $\mathbb{C}$-scheme with some special involution. Here the *cover* taken is the Galois cover $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ and the involution property has to do with the fact that the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is of order 2.

(2) The *descended* $\mathbb{R}$-scheme that we get is a $\mathbb{R}$-conic (curve described by a degree 2 equation in $\mathbb{P}^2_{\mathbb{R}}$) which has the strange property that it has no $\mathbb{R}$-points. In general if k is a field and $C \subset \mathbb{P}^2_k$ is a regular conic with admits at least one k-rational point (which here is not the case for $\mathbb{P}^1_{\mathrm{tw}}$), then it is actually isomorphic to $\mathbb{P}^1_k$. This can be understood *via* the nice method of stereographic projection. See here for for a nice explanation of this method.

(3) The $\mathbb{R}$-scheme $\mathbb{P}^1_{\mathrm{tw}}$ appears in cohomology theory. Namely one can show that vector bundles on this scheme are intimately related to *Hodge structures.* Singular cohomology groups of projective varieties admits Hodge structures, that's why we care about this kind of linear algebraic objects.

Solution key. (1) *(Claudio)* Recall that the $\mathbb{C}$-rational points of $\mathbb{P}^1_{\mathbb{C}}$ given by the homogeneous ideals $(\beta x - \alpha y)$ are identified with elements $[\alpha : \beta]$ of the classical projective space. This was part of the last homework exercise, to identify $\mathbb{C}$-rational points we use exercise 4 of sheet 3 and the local charts $D_+(x)$ and $D_+(y)$ of $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x, y])$. Let's see where $\tau : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ takes $(\beta x - \alpha y)$. Recall that

$$\tau((\beta x - \alpha y)) = \sigma^{-1}((\beta x - \alpha y)) = (\overline{\beta} y + \overline{\alpha} x)$$

where we used that $\sigma : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ is invertible with inverse given by

$$\sigma^{-1} : \mathbb{C}[x, y] \to \mathbb{C}[x, y] : f(x, y) \mapsto \overline{f}(y, -x).$$

Hence $[\alpha : \beta]$ is sent to $[\overline{\beta} : -\overline{\alpha}]$ under the corresponding identifications which furthermore means that τ carries $[1 : 0]$ to $[0 : -1] = [0 : 1]$ and thus $\infty \mapsto 0$, $[z : 1]$ is carried to $[1 : -\overline{z}] = [-\frac{1}{\overline{z}} : 1]$ if $z \neq 0$ and thus $z \mapsto -\frac{1}{\overline{z}}$ and finally $[0 : 1]$ is carried to $[1 : 0]$ which implies that $0 \mapsto \infty$.

This concludes the first part of the exercise.

(2) *(Céline)* Notice that $\sigma^2(x) = -x$. Hence, $\tau^2 \mid_{D_+(x)}: D_+(x) \to D_+(x)$. Recall that $D_+(x) \cong \mathrm{Spec}(\mathbb{C}[\frac{y}{x}])$. Now, since we have an equivalence of categories between rings and affine schemes, the map $\tau^2 : \mathrm{Spec}(\mathbb{C}[\frac{y}{x}]) \to \mathrm{Spec}(\mathbb{C}[\frac{y}{x}])$ corresponds to a ring homomorphism $\mathbb{C}[\frac{y}{x}] \to \mathbb{C}[\frac{y}{x}]$. We find that this map is

$$\begin{array}{ccc} \mathbb{C}[\frac{y}{x}] & \longrightarrow & \mathbb{C}[\frac{y}{x}] \\ \frac{y}{x} & \longmapsto & \frac{\sigma^2(y)}{\sigma^2(x)} = \frac{-y}{-x} = \frac{y}{x} \end{array}$$

which is the identity. Thus, $\tau^2 \mid_{D_+(x)}: D_+(x) \to D_+(x)$ is the identity. Similarly, $\tau^2 \mid_{D_+(y)}: D_+(y) \to D_+(y)$ is the identity. Hence, by the gluing property, $\tau^2 = \mathrm{id}$.

(3) *(Benoît)* We want to show that the $\mathbb{R}$-algebra of fixed points $\mathbb{C}[x, y]^\sigma$ is given by $\mathbb{R}[ixy, x^2 + y^2, i(x^2 - y^2)]$. Denote this ring by A, and note that as an $\mathbb{R}$-algebra, $A = (1, ixy, x^2 + y^2, i(x^2 - y^2))$. Clearly, all four generators of A are fixed by σ, so that we only have to show that $\mathbb{C}[x, y]^\sigma \subseteq A$. Let $f \in \mathbb{C}[x, y]^\sigma$, if f is constant, $f = \bar{f}$ by assumption, so we are done. If f is not constant, we may assume that it is homgeneous. As seen already, if f is of degree d, we have

$\sigma^2(f) = (-1)^d f$, so d is necessarily even. We first show the claim for f of the form $ax^{2n} + \overline{a}y^{2n}$ by induction. When $n = 0$, f is constant and we are done, and when $n = 1$, we have that $f = \mathrm{Re}(a)(x^2 + y^2) + \mathrm{Im}(a)i(x^2 - y^2)$ so that $f \in A$. Now suppose the claim holds for even degree lower than $2n$, and note that :

$$ax^{2n}+\overline{a}y^{2n}-(x^2+y^2)(ax^{2n-2}+\overline{a}y^{2n-2}) = -ax^{2n-2}y^2-\overline{a}x^2y^{2n-2} = (ixy)^2(ax^{2n-4}+\overline{a}y^{2n-4}),$$

so that indeed $f \in A$ by induction hypothesis. Now returning to the general case, note that for f any homogeneous element of degree $2n$, we can write :

$$f(x,y) = \sum_{i+j=2n} a_{i,j}x^iy^j = a_{2n,0}x^{2n}+a_{0,2n}y^{2n}-ixy\sum_{\substack{i+j=2n\\ i>0,j>0}} ia_{i,j}x^{i-1}y^{j-1} =: g(x,y)-ixy\tilde{f}(x,y).$$

As f is a fixed point, we see that $a_{2n,0} = \overline{a_{0,2n}}$ so that $g \in A$ by previous part. Moreover, $\tilde{f} = -\frac{f-g}{ixy}$ is also fixed by sigma, so that it is a form of even degree $\leq 2n-2$. Iterating this process, we conclude that f is indeed in A.

(4) *(Céline)* Notice that, from the previous point, $\mathbb{C}[x,y]^\sigma$ is an $\mathbb{R}$-algebra. Moreover, $\mathbb{C} \cong \mathbb{R}[i]$ is an $\mathbb{N}$-graded ring such that the set of homogeneous elements of degree 0 of $\mathbb{C}$ is an $\mathbb{R}$-algebra. By exercise 2 of exercise sheet 5, we have that tw $\times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong \mathrm{Proj}(\mathbb{C}[x,y]^\sigma \otimes_{\mathbb{R}} \mathbb{C})$. Since $\mathbb{C}[x,y]^\sigma = \mathbb{R}[ixy, x^2+y^2, i(x^2-y^2)]$, we have that $\mathbb{C}[x,y]^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[ixy, x^2+y^2, i(x^2-y^2)] = \mathbb{C}[x^2,y^2,xy]$. Then, using exercise 5 of exercise sheet 4, we find that $\mathrm{Proj}(\mathbb{C}[x^2,y^2,xy]) \cong \mathrm{Proj}(\mathbb{C}[x,y]) = ¶^1_{\mathbb{C}}$.

(5) *(Benoît)* Let ψ be the 2-homogeneous map of $\mathbb{R}$-algebras $\mathbb{R}[t_1,t_2,t_3] \to \mathbb{C}[x,y]^\sigma$ sending

$$t_1 \mapsto 2ixy, \quad t_2 \mapsto x^2+y^2, \quad t_3 \mapsto i(x^2-y^2.)$$

By point (3), ψ is clearly surjective, let us identify its kernel. We see that $\ker\psi$ contains the principal ideal $I = (f) := (t_1^2+t_2^2+t_3^2)$. Moreover, I is prime, as for example $\mathrm{ev}_{t_2=1,t_3=0}(f) = t_1^2+1$ is irreducible, implying that f itself is irreducible. To show that $\ker\psi = I$, we use an argument dimension. Indeed, we have by a result of the course that $\mathrm{ht}(\ker\psi) = \dim(\mathbb{R}[t_1,t_2,t_3]) - \dim(\mathbb{C}[x,y]^\sigma)$ by surjectivity. To compute $\dim(\mathbb{C}[x,y]^\sigma)$, we use the following fact : *Let $k \subseteq k'$ be an extension of fields, and A a finitely generated k algebra. Then*

$$\dim(A \otimes_k k') = \dim(A).$$

Proof. Let $d = \dim(A)$, By the Noether normalization lemma, there exists an integral extension $k[x_1,...,x_d] \hookrightarrow A$ for some $x_1,...,x_d$. Tensoring with k' preserves injectivity (as k' is free, it is in particular flat). Moreover, the extension $k'[x_1,...,x_d] \hookrightarrow A \otimes_k k'$ is still integral (as it is still finite). We deduce that $\dim(A \otimes_k k') = \dim_(k'[x_1,...,x_d]) = d$ as wanted. □

By the lemma, the dimension $\mathbb{C}[x,y]^\sigma$ is the same as the dimension of $\mathbb{C}[x^2,xy,y^2]$, which is 2, since it admits $\mathbb{C}[x^2,y^2]$ as a Noether normalization for example. Therefore, $\operatorname{ht}(\ker\psi) = 1$, which implies that $\ker\psi = I$.

(6) *(Benoît)* We want to show that the twistor has no $\mathbb{R}$-rational points. Note that by point (3), we have induced map of $\mathbb{R}$-schemes $\mathrm{tw} \to \mathbb{P}^2_{\mathbb{R}}$ (by exercise 5.4 sheet 4 for example). Moreover, this is a closed embedding onto $V_+(t_1^2 + t_2^3 + t_3^2)$. By way of contradiction, suppose we have an $\mathbb{R}$-point in the twistor, this gives an $\mathbb{R}$-point of $\mathbb{P}^2_{\mathbb{R}}$ via this map. Now $\mathbb{P}^2_{\mathbb{R}}$ is covered by $D_+(t_i)$, so without loss generality, let us assume our $\mathbb{R}$-point lands in $D_+(t_3)$. Therefore, it corresponds to a prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathbb{R}[u, v])$ which contains $(u^2 + v^2 + 1)$, where $u = \frac{t_1}{t_3}, v = \frac{t_2}{t_3}$. Then by hypothesis on $\mathfrak{p}$, we have an isomorphism

$$\left(\mathbb{R}[u, v] \big/ \mathfrak{p} \right)_{\mathfrak{p}} \cong \mathbb{R},$$

but this is a contradiction, as the image $a, b \in \mathbb{R}$ of u, v must satisfy $a^2 + b^2 = 1$.

□