Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

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AG II - Schemes and sheaves

Solutions

Exercise 1. Let $\varphi\colon A \to B$ be a ring map, and let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the associated morphism of schemes.

(1) Let $\operatorname{nil}(A)$ denote the nilpotent elements in A. Use the equality
$$\operatorname{nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$$
to show, as a consequence of the definitions, that $D(a) = \emptyset$ if and only a is nilpotent.

(2) Show that if φ is injective, then f has dense topological image.

(3) Further assume that A is reduced. Show that, if f has dense topological image, then φ is injective.

Solution. (1) Fix $a \in A$ By definition, $D(a)$ consists of the primes in A that do not contain a. Now, if $a \in \operatorname{nil}(A)$, then $a \in \mathfrak{p}$ for every $\mathfrak{p}$, and hence $D(a) = \emptyset$. Conversely, assume that a is not nilpotent. Then, by the description of $\operatorname{nil}(A)$ provided, there exists a prime $\mathfrak{p}$ such that $a \notin \mathfrak{p}$. In particular, $\mathfrak{p} \in D(a)$ holds, and the claim follows.

(2) Let $a \in A$ such that $D(a)$ is not-empty, meaning that a is not nilpotent. Because φ is injective, $\varphi(a)$ is also not nilpotent. Therefore $D(\varphi(a)) = f^{-1}(D(a)) \neq \emptyset$, which concludes.

(3) Let $a \in \ker(\varphi)$. Therefore $f^{-1}(D(a)) = \emptyset$. Because we supposed that the image is dense, the only possibility is that $D(a) = \emptyset$, meaning that a is nilpotent. Because we supposed that A is reduced, we conclude that φ is injective, a being forced to be zero.

□

Exercise 2. Let k be an algebraically closed field. Let $R = k[x, y, z]/(xy - z^2)$ and $X = \mathrm{Spec}(R)$. In the following, the notation $\overline{x}$, etc., denotes the image of x, etc., in the quotient ring $k[x, y, z]/(xy - z^2)$.
In the exercise, you can freely use the following formulation of the Jacobian criterion:
Let $A = k[x_1, \ldots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of f with respect to x_i. Then

$$\mathrm{Spec}(A) \text{ is regular } \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

Moreover the closed subset $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of $\mathrm{Spec}(A)$*.*

(1) Is X regular? Answer this question using the Jacobian criterion and list (if any) all the non-regular k-points.
(2) Prove that X is normal.
In the following, identify R with $k[u^2, v^2, uv]$ under the identification $\overline{x} \mapsto u^2$, $\overline{y} \mapsto v^2$, $\overline{z} \mapsto uv$, where u and v are variables. You can use this identification without proof. In the following, we write $\mathbb{A}^2_k = \mathrm{Spec}(k[u, v])$. Let $f \colon \mathbb{A}^2_k \to X$ be the morphism given by the inclusion $k[u^2, v^2, uv] \hookrightarrow k[u, v]$.
(3) Show that f is a finite morphism.
(4) Compute the fibers of f over the points (u^2, v^2, uv) and $(u^2 - 1, v^2 - 1, uv - 1)$. For each of them, determine whether the fiber is reduced and provide its cardinality.

Warning: In the following, some of the answers may depend on $\mathrm{char}(k)$. Watch out!

Solution. In the following, we freely use that $k[x, y, z]/(xy - z^2)$ is a domain and a finitely generated k-algebra. In particular, additivity of height and dimension holds, and so does Serre's criterion for normality.

(1) X is not regular. Since k is algebraically closed, we can apply the Jacobian criterion to determine all non-regular k-points. The gradient of the polynomial $xy - z^2$ is $\langle y, x, -2z\rangle$. The common zeroes of $xy - z^2$ and $\langle y, x, -2z\rangle$ consist of only the point with Cartesian coordinates $(0, 0, 0)$.
(2) We apply Serre's criterion for normality. By the Jacobian criterion, we know that the non-regular locus is the closed set $V(xy - z^2, y, x, -2z)$, which by the previous part only contains a closed point, which, by additivity, corresponds to a prime of height 2. In particular, X has the property R_1. To conclude, we only need to check X satisfies the S_2 property. At a regular point of codimension k we have a regular sequence of length exactly k. Thus, we only need to check the S_2 property at the only singular point. The point (x, y, z) (which corresponds to the Cartesian point $(0, 0, 0)$) corresponds to a prime of height 2. Then, we consider x, y as a regular sequence, and we conclude that X is normal.
Alternatively, we include a direct approach as well. By worksheet 6, it is enough to show that the domain R is integrally closed. We

use the identification with $k[u^2, v^2, uv]$. Namely, the ring is the subring of $k[u, v]$ whose elements are sums of monomials of even degree. Now, we use that $k[u, v]$ is integrally closed (this fact was discussed in class). Thus, any element that is integral over R is an element of $k[u, v]$. But any such element $s(u, v)$ can be also written as a ratio of polynomials $p(u, v)/q(u, v)$, where all the monomials in p and q have even degree. Then, one can deduce that all the monomials in $s(u, v)$ need to have even degree. Indeed $s(u, v)q(u, v) = p(u, v)$ so if by contradiction there is a monomial of odd degree in $s(u, v)$, as a monomial of odd degree times a monomial of even degree is of odd degree, we would get a contradiction.

(3) The inclusion realizes $k[u, v]$ as a $k[u^2, v^2, uv]$-module. We claim that the elements $\{1, u, v\}$ generate $k[u, v]$ as a $k[u^2, v^2, uv]$-module. If so, then the desired assertion follows from the claim by the definition of a finite morphism. To show the claim, it suffices to show that every monomial in u and v is generated over $k[u^2, v^2, uv]$ by $1, u, v$. This is clear by direct inspection.

(4) By definition of fiber and the fact that localization commutes with quotients, we need to compute the following tensor products:
   (a) $k[u, v] \otimes_{k[u^2,v^2,uv]} k[u^2, v^2, uv]/(u^2, v^2, uv) \simeq k[u, v]/(u^2, v^2, uv) = k[u, v]/(u, v)^2$. Thus, the desired fiber is $\mathrm{Spec}(k[u, v]/(u, v)^2)$. In particular, the underlying topological space of the fiber is the same space as the one obtained by quotienting by the maximal ideal (u, v). In particular, the scheme we obtain consists of only one point (whose Cartesian coordinates are $(0, 0)$), and such scheme is not reduced; and
   (b) $k[u, v] \otimes_{k[u^2,v^2,uv]} k[u^2, v^2, uv]/(u^2 - 1, v^2 - 1, uv - 1) \simeq k[u, v]/(u^2 - 1, v^2 - 1, uv - 1)$. In this ring, $\overline{u}$ and $\overline{v}$ (i.e., the images of u, v in the quotient ring) are units, so, by taking linear combinations of the generators and factoring out the units $\overline{u}$ or $\overline{v}$, we see that the relation $u = v$ is satisfied. But then, it is equivalent to quotienting by the product ideal $(u - 1, v - 1)(u + 1, v + 1)$. In particular, if the characteristic of k is not 2, the underlying scheme is reduced and it is the disjoint union of two closed points, corresponding to points with Cartesian coordinates $(1, 1)$ and $(-1, -1)$. If the characteristic is 2, we obtain one non-reduced point.
   Alternatively, we can also argue by considering the following chain of isomorphisms:

$$\frac{k[u, v]}{(uv - 1)}/(uv - 1, u^2 - 1, v^2 - 1) \cong \frac{k[u, 1/u]}{(u^2 - 1, 1/u^2 - 1)} \cong \frac{k[u]}{(u^2 - 1)}.$$

□

Exercise 3. Let k be an algebraically closed field. Let $R = k[x,y,z]/(xy - z^2)$ and $X = \operatorname{Spec}(R)$. In the following, the notation $\overline{x}$, etc., denotes the image of x, etc., in the quotient ring $k[x,y,z]/(xy - z^2)$.
In the following, you may (it is not necessary, but you may find it useful for the computations) identify R with $k[u^2, v^2, uv]$ under the identification $\overline{x} \mapsto u^2$, $\overline{y} \mapsto y^2$, $\overline{z} \mapsto uv$ where u and v are variables. You can use this identification without proof.

(1) Show that the ideal $(\overline{x}, \overline{z})$ of R is prime in R and that $\operatorname{Spec}(R/(\overline{x}, \overline{z}))$ is isomorphic to $\mathbb{A}^1_k$.
(2) Let P be the prime Weil divisor corresponding to $(\overline{x}, \overline{z})$ and let D be the Weil divisor corresponding to the Cartier divisor $(\overline{x})$. Show that $D = 2 \cdot P$.
Note: You can utilize without proof that X is normal, and, in particular, regular in codimension 1 (cf., Exercise 2). In particular, the theories of Weil and Cartier divisors are well posed.

Solution. (1) To show primality, we argue that the quotient is an integral domain. By the third isomorphism theorem, we have $R/(\overline{x}, \overline{z}) \simeq k[x,y,z]/(x,z) \simeq k[y]$, which is an integral domain. Furthermore, this is a polynomial ring in one variable, so also the second claim follows.

(2) For ease with computations, we utilize the identification provided as hint. First, we observe that (u^2) is a principal ideal, so we indeed obtain a Cartier divisor. Also, observe that, if $u^2 = 0$, then we have $(uv)^2 = u^2v^2 = 0$. Thus, we have $V(u^2) = V(u^2, uv)$. In particular, it follows that D is an integral multiple of P, since the only codimension 1 point where the regular function u^2 vanishes is the generic point of P. To obtain the coefficient of proportionality, we need to compute the order of vanishing at such generic point. Thus, we need to localize our algebra at the prime (u^2, uv). We observe that v^2 is in the complement of (u^2, uv) and thus is a unit in this localization. Thus, the ratio $uv/v^2 = u/v$ is an element in this localization. Furthermore, we have $u^2 = (u/v) \cdot uv$. Thus, in this localization, the ideals (u^2, uv) and (uv) are the same ideal. By assumption, we know that X is regular at the generic point of P. In particular, we obtain a DVR with maximal ideal $(u^2, uv) = (uv)$. Thus, uv is a local uniformizer of this DVR. Thus, we need to check the divisibility of u^2 by the local uniformizer uv. We have $u^2 = (uv)^2/v^2$, where $1/v^2$ is a unit. In particular, u^2 vanishes of order 2 at the generic point of P, and hence $D = 2 \cdot P$ holds.

□

Exercise 4. Let k be a field, set $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, x_1, \dots, x_n])$. Recall that, for an $\mathcal{O}_{\mathbb{P}^n_k}$-module $\mathcal{F}$ and any integer m, $\mathcal{F}(m)$ denotes $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(m)$.

(1) Show that for a coherent $\mathcal{O}_{\mathbb{P}^n_k}$-module $\mathcal{F}$, there is a short exact sequence,

$$0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{E}$ is a finite direct sum of copies of $\mathcal{O}_{\mathbb{P}^n}(l)$ for some $l \in \mathbb{Z}$ and $\mathcal{N}$ is coherent.

(2) Recall that the canonical isomorphism of $\mathcal{O}_{\mathbb{P}^n_k}$-modules $\mathcal{F}(m) \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(j) \cong \mathcal{F}(m+j)$ makes the k-vector space

$$\bigoplus_{j \in \mathbb{N}} \Gamma(\mathbb{P}^n, \mathcal{F}(j))$$

a graded module over the graded ring $\underset{j\in\mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(j)) \cong k[x_0, x_1, \dots, x_n]$. Show that, when $\mathcal{F}$ is coherent, $\underset{j\in\mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n_k, \mathcal{F}(j))$ is a finitely generated module over $k[x_0, x_1, \dots, x_n]$.

Hint: *Use part (1) to relate* $\underset{j\in\mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n, \mathcal{F}(j))$ *and* $\underset{j\in\mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n, \mathcal{E}(j))$.

Solution. (1) Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, there exists a natural number m such that $\mathcal{F}(m)$ is globally generated. Recall that $\Gamma(\mathbb{P}^n, \mathcal{F}(m))$ is a finite dimensional k-vector space. Choose a basis $e_1, e_1, \dots, e_d$ of $\Gamma(\mathbb{P}^n, \mathcal{F}(m))$. Consider the $\mathcal{O}_{\mathbb{P}^n}$-linear map from a direct sum of d-copies of $\mathcal{O}_{\mathbb{P}^n}$ to $\mathcal{F}(m))$, which sends the $1 \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ of the i-th copy to $e_i \in \Gamma(\mathbb{P}^n, \mathcal{F}(m))$, for $1 \le i \le d$:

$$\bigoplus^d \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(m).$$

Since $\mathcal{F}(m)$ is globally generated and $e_1, e_2, \dots, e_d$ is a k-basis of the space of global sections of $\mathcal{F}(m)$, the above map is sujective; denote the kernel by $\mathcal{N}'$. Since the kernel of a map between two coherent sheaves is coherent, $\mathcal{N}'$ is coherent. Tensoring the resulting exact sequence:

$$0 \to \mathcal{N}' \to \bigoplus^d \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(m) \to 0,$$

by $\mathcal{O}(-m)$, we get an exact sequence:

$$0 \to \mathcal{N}' \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(-m) \to \bigoplus^d \mathcal{O}_{\mathbb{P}^n}(-m) \to \mathcal{F} \to 0.$$

Here, $\otimes \mathcal{O}(-m)$ preserves exactness as $\mathcal{O}(-m)$ is locally free. Since tensor product of two coherent sheaves is coherent, $\mathcal{N}' \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(-m)$ is coherent. Therefore, the last sequence is a desired one.

(2) Fix an exact sequence as in part (1). For each natural number j, tensoring this exact sequence with $\mathcal{O}(j)$, we get an exact sequence

of sheaves:

$$0 \to \mathcal{N}(j) \to \mathcal{E}(j) \to \mathcal{F}(j) \to 0$$

The induced long exact sequence of sheaf cohomology groups yields an exact sequence:

$$0 \to \Gamma(\mathbb{P}^n, \mathcal{N}(j)) \to \Gamma(\mathbb{P}^n, \mathcal{E}(j)) \to \Gamma(\mathbb{P}^n, \mathcal{F}(j)) \to \mathrm{H}^1(\mathbb{P}^n, \mathcal{N}(j))$$

Taking direct sum gives another exact sequence of graded $k[x_0, \ldots, x_n]$ modules:

$$\bigoplus_{j \in \mathbb{N}} \Gamma(\mathbb{P}^n, \mathcal{E}(j)) \to \bigoplus_{j \in \mathbb{N}} \Gamma(\mathbb{P}^n, \mathcal{F}(j)) \to \bigoplus_{j \in \mathbb{N}} \mathrm{H}^1(\mathbb{P}^n, \mathcal{N}(j)).$$

By Serre's criterion for ampleness, there exists an integer j_0 such that, for all $j \geq j_0$, $\mathrm{H}^1(\mathbb{P}^n, \mathcal{N}(j)) = 0$. Since for each j, $\mathrm{H}^1(\mathbb{P}^n, \mathcal{N}(j))$ is a finite dimensional k-vector space, the $k[x_0, \ldots, x_n]$-module $\underset{j \in \mathbb{N}}{\oplus} \mathrm{H}^1(\mathbb{P}^n, \mathcal{N}(j))$ is finitely generated. We claim that $\underset{j \in \mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n, \mathcal{E}(j))$ is a finitely generated $k[x_0, \ldots, x_n]$-module. Indeed, for a fixed integer m,

$$\bigoplus_{j \in \mathbb{N}} \Gamma(\mathbb{P}^n, \mathcal{O}(m)(j))$$

is isomorphic to $k[x_0, \ldots, x_n](m)$. Since $\mathcal{E}$ is a direct sum of finitely many copies of possibly different $\mathcal{O}(k)$'s and a finite direct sum of finitely generated modules is finitely generated, $\underset{j \in \mathbb{N}}{\oplus} \Gamma(\mathbb{P}^n, \mathcal{E}(j))$ is finitely generated. Now the desired finite generation follows from the fact: given an exact sequence of modules over a noetherian ring

$$M' \to M \to M'',$$

if both M' and M'' are finitely generated as modules, so is M.

$\square$

Exercise 5. Let S, X, Y be schemes. Let $f\colon X \to Y$ be a map of S-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph map Γ_f defined using the universal property of the product of S-schemes

$$\Gamma_f := (\mathrm{id}, f)\colon X \to X \times_S Y$$

is a closed immersion.
Hint: *Use that the diagonal map* $(\mathrm{id}, \mathrm{id})\colon Y \to Y \times_S Y$ *is a closed immersion by definition of separatedness, and that closed immersions are stable under pullback.*

(2) Suppose now that X is proper over S. Show that f is closed, meaning that the underlying map of topological spaces is a closed map.

Solution. (1) We claim that

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \Gamma_f}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_S Y & \xrightarrow{f \times \mathrm{id}} & Y \times_S Y \end{array}$$

is a pullback square. This is checked by universal property: a pair of maps $(a, (b, c))$ going to Y and $X \times_S Y$ creating a commutating square with the above will satisfy $a = c = fb$. Indeed using the definitions of the maps we get that commuting means $(a, a) = (bf, c)$. Therefore there is unique map to X that makes the diagram commute, this map being b. This by definition shows that the square is Cartesian. Now, because $\Delta\colon Y \to Y \times_S Y$ being a closed immersion by separatedness, and that closed immersions are stable by pullbacks, we conclude.

(2) As $X \to X \times_S Y \to Y$ is closed as a composition of closed maps, the first map being closed by the first point and the second map being closed by universal closedness of $X \to S$.

□

Exercise 6. Let k be an algebraically closed field. Let X be an integral (i.e., reduced and irreducible), regular, one dimensional projective scheme over k and x be a closed point of X.

(1) Since the local ring $\mathcal{O}_{X,x}$ is regular, the unique maximal ideal of $\mathcal{O}_{X,x}$ is generated by one element. Fix a generator, say t, of the maximal ideal. Show that the element $d_{\mathcal{O}_{X,x}/k}(t)$ generates the stalk $(\Omega_{X/k})_x$ as an $\mathcal{O}_{X,x}$-module.

(2) Show that the $\mathcal{O}_{X,x}$-linear map $\mathcal{O}_{X,x} \to (\Omega_{X/k})_x$ sending 1 to $d_{\mathcal{O}_{X,x}/k}(t)$ is an isomorphism of $\mathcal{O}_{X,x}$-modules.

(3) Given $g \in \mathcal{O}_{X,x}$ and t as above, by part (2), there is a unique $h \in \mathcal{O}_{X,x}$ such that

$$d_{\mathcal{O}_{X,x}/k}(g) = h d_{\mathcal{O}_{X,x}/k}(t).$$

Denote such an h by dg/dt. Show that the map $g \mapsto dg/dt$ is a k-linear derivation of $\mathcal{O}_{X,x}$.

(4) Show that given any other k-linear derivation $\delta \in \mathrm{Der}_k(\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$, there is a unique $f_\delta \in \mathcal{O}_{X,x}$, such that for all $g \in \mathcal{O}_{X,x}$,

$$\delta(g) = f_\delta dg/dt.$$

Solution. (1) Denote the maximal ideal of $\mathcal{O}_{X,x}$ by m_x. Since x is a closed point and k is algebraically closed, the composition

$$k \to \mathcal{O}_{X,x} \to \frac{\mathcal{O}_{X,x}}{m_x}$$

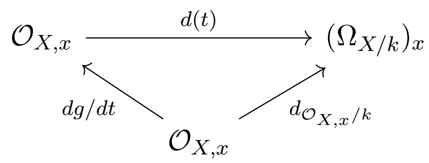
is an isomorphism. Therefore we know that the canonical map

$$\frac{m_x}{m_x^2} \to (\Omega_{X/k})_x \otimes \frac{\mathcal{O}_{X,x}}{m_x},$$

sending $a \pmod{m_x^2}$ to $d_{\mathcal{O}_{X,x}/k}(a) \pmod{m_x}$ is an isomorphism. So $d_{\mathcal{O}_{X,x}/k}(t) \pmod{m_x}$ generates $(\Omega_{X/k})_x/m_x(\Omega_{X/k})_x$. Since X is finite type over k, the $\mathcal{O}_X$-module $\Omega_{X/k}$ is coherent. Thus $(\Omega_{X/k})_x$ is a finitely generated module over the noetherian ring $\mathcal{O}_{X,x}$. Thus by Nakayama's lemma, $d_{\mathcal{O}_{X,x}/k}(t)$ generates $(\Omega_{X/k})_x$.

(2) The given map is surjective by part (1). Denote the kernel of this map by I. So $(\Omega_{X/k})_x$ as an $\mathcal{O}_{X,x}$-module is isomorphic to $\mathcal{O}_{X,x}/I$. On the other hand, since X is regular, $\Omega_{X/k}$ is locally free of rank one. So $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$-module of rank one. In particular, $(\Omega_{X/k})_x$ is a torsion free $\mathcal{O}_{X,x}$-module. Therefore $I = 0$. So the given map is an isomorphism.

(3) Note that the following diagram commutes:



where the top arrow is multiplication by $d_{\mathcal{O}_{X,x}/k}(t)$. Since by part (2), the top arrow is an isomorphism of $\mathcal{O}_{X,x}$-modules, the top left copy of $\mathcal{O}_{X,x}$ serves as the module of Kähler differentials with the universal k-linear derivation d/dt, where $d/dt(g) = dg/dt$.

(4) By part (2), $\mathcal{O}_{X,x}$ is the module of Kähler differentials of the k-algebra $\mathcal{O}_{X,x}$, and, by part (3), d/dt serves as universal derivation. So, given any such derivation δ, there exists a unique $\mathcal{O}_{X,x}$-linear endomorphism of $\mathcal{O}_{X,x}$, say ϕ, such that $\delta = \phi d/dt$. But any such ϕ is multiplication by f_δ, where $f_\delta = \phi(1)$.

□