Modern algebraic geometry (MATH-510) — Final exam

, 8 h 15 – 11 h 15

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Nom :

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SCIPER :

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Paper & pen: This booklet contains 5 exercises, on 6 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Open book exam & results of the course: This is an open book exam. That is, you can have any printed or handwritten help with you, but you cannot use electronic devices.

In your solutions, you can use all the material learned during the lectures and the exercise sessions, including the lecture notes, Hartshorne, and the solutions of the exercise sheets. However, please state always what you are using. If you are using a not very frequently used statement, please give a precise reference, preferably with numbers.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: all rings are commutative and with identity.

Exercice 1 [**10 pts**]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let k be a field. Let $X = \operatorname{Spec}\left(k[x,y] \big/ (xy^3) \right)$.

(1) Show that the image of xy via the localization homomorphism

$$k[x,y] \big/ (xy^3) \to \left(k[x,y] \big/ (xy^3) \right)_{(x,y)} \cong k[x,y]_{(x,y)} \big/ (xy^3)$$

is non-zero.

(2) Determine if $\mathcal{O}_{X,P}$ is reduced or not for $P = (x,y) \in X$, and prove your statement.

Exercice 2 [**20 pts**]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let X be a Noetherian scheme, and let $|X|$ be the underlying topological space of X.

(1) Show that $|X|$ is discrete (i.e., all subsets are open) if and only if $\dim X = 0$.

(2) Show that $\dim X = \dim X_{\text{red}}$.

(3) Show that if all stalks of $\mathcal{O}_X$ are finite, then $\dim X = 0$.

You can use without proof the (not too hard) consequence of the Rings and modules material that a ring is Artinian if and only if it is Notherian and of dimension 0.

Exercice 3 [**15 pts**]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let R be a ring. An affine conic over R is a scheme of the form $X = \operatorname{Spec}\left(R[x,y] \big/ (f) \right)$, where $\deg f = 2$. Note that X comes with a structure morphism $\lambda : X \to \operatorname{Spec} R$ given by the embedding of R into $R[x,y]$. A section of X is a morphism $\sigma : \operatorname{Spec} R \to X$ such that $\lambda \circ \sigma = \mathrm{id}_{\operatorname{Spec} R}$.

(1) Show that sections of X are in one to one correspondence with pairs $(x_0, y_0) \in R^{\oplus 2}$ such that $f(x_0, y_0) = 0$.

(2) What is another name of a section of X in the case when R is a field?

(3) Show that if $R = k$ is an algebraically closed field then there is a section.

(4) For $R = \mathbb{Z}$, give examples of f when X has a section, and also when X does not have a section.

Exercice 4 [**20 pts**]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let k be a field and let X be a projective scheme over k. Let $\overline{k}$ be an algebraic closure of k. We denote the base change to the algebraic closure by $X_{\overline{k}} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Recall that X is a *geometrically connected* (resp. *geometrically irreducible*) k-scheme if $X_{\overline{k}}$ is connected (resp. irreducible).

To save time, accept the following statement without proof (this statement is not too hard to prove from what we have learned): $H^0(X, \mathcal{O}_X) = \prod_{i=1}^N R_i$, where R_i are Artinian local rings, and N is the number of connected components of X.

(1) Show that $H^0\left(X_{\overline{k}}, \mathcal{O}_{\overline{k}}\right) \cong H^0(X, \mathcal{O}_X) \otimes_k \overline{k}$.

Hint: Recall that $\overline{k}$ is flat over k. Also, the Čech description of $H^0(X, \mathcal{O}_X)$ might be useful here.

(2) Show that if $H^0(X, \mathcal{O}_X) = k$, then X is geometrically connected.

(3) Show that $X = \operatorname{Proj}\left(\mathbb{R}[x,y,z] \big/ (x^2 + y^2)\right)$ over $k = \mathbb{R}$ is an integral, geometrically connected k-scheme, which is not geometrically irreducible.

Hint: you can use without proof the commutative algebra statements that if a ring R is a domain, than so are all its localizations and the same statement for domain replaced by being reduced (both are straight forward consequences of the definitions, we are just saving time with assuming them).

Exercice 5 [**35 pts**]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let k be a field and let $X := \mathbb{P}^1_k = \operatorname{Proj} k[x, y]$. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module of rank 2.

(1) Show that for any locally free sheaf $\mathcal{F}$ on X and for any $a \in \mathbb{Z}$, we have

$$\operatorname{Hom}_X\big(\mathcal{O}_X(a), \mathcal{F}(a)\big) \cong \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{F}) \cong H^0(X, \mathcal{F}).$$

(2) Show that for any locally free sheaf $\mathcal{F}$ on X, for any $a \in \mathbb{Z}$, and for any non-zero $\mathcal{O}_X$-module homomorphism $\phi : \mathcal{O}_X(a) \to \mathcal{F}$, the homomorphism ϕ is in fact injective.

Hint: 1.) show that ϕ is injective at the generic point. 2.) show that it is injective on a non-zero open set, 3.) consider $\ker\phi$*, and use that it is a subsheaf of $\mathcal{O}_X(a)$.*

(3) For any integer $a \in \mathbb{Z}$, show that there in a one-to-one correspondence between $H^0\big(X, \mathcal{E}(-a)\big)$ and embeddings $\mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{E}$. Deduce that there exists $a \in \mathbb{Z}$ for which an embedding $\mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{E}$ exists, and additionally show that there is a maximal such a.

(4) Show that if $\mathcal{F}$ is not a locally-free coherent sheaf on X, then $H^0(X, \mathcal{F}) \neq 0$.

Hint: you may want to use the Fundamental theorem of finitely generated modules over a PID.

(5) Let us call $a_{\max}$ the maximal a found above. Show that $\mathcal{E}/\mathcal{O}_{\mathbb{P}^1_k}(a_{\max})$ is an invertible sheaf (that is, the whole Exercise 5 shows that $\mathcal{E}$ is an extension of two invertible sheaves).

Hint: note that $\left(\mathcal{E}/\mathcal{O}_{\mathbb{P}^1_k}(a_{\max})\right)(-a_{\max} - 1) \cong \mathcal{E}(-a_{\max} - 1)/\mathcal{O}_{\mathbb{P}^1_k}(-1)$.