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AG II - Schemes and sheaves

Solutions – week 1

Exercise 1. *Refresh*
The goal of this exercise is to refresh some notions of commutative algebra as well as their interpretation in algebraic geometry. Let k be an algebraically closed field.

(1) Recall (your definition) of $\mathbb{A}^n_k$ and that the polynomnial algebra $k[x_1, \ldots, x_n]$ is to be interpreted as *functions* on this space.
(2) A *finite type k-algebra A* is a k-algebra who admits a surjection $f : k[x_1 \ldots, x_n] \to A$. The algebra A is to be interpreted as functions on which space ? What is the interpretation of the ideal $\ker(f)$?
(3) Recall what the *localization of a ring on a multiplicative subset* is. Recall that this is an exact functor. Recall the important example of the localization at a prime ideal of a ring.
(4) Let A be a finite type k-algebra. Let $a \in A$. Recall the localization A_a (so with respect to the multiplicative subset $\{a^n\}_{n\geq 0}$). The ring A_a is to be interpreted as functions on which space ? Is $k[x, y]_y$ a finite type k-algebra ?
(5) Let R be a ring R' and R'' some R-algebras. Recall what is the *tensor product* $R' \otimes_R R''$. What is the R-algebra law on this tensor product ? Which is the universal property of this object as an R-algebra ?
(6) Recall that for a ring R, an ideal I of R, multiplicative subset S of R and an R-algebra $\varphi : R \to A$,

$$R/I \otimes_R A \cong A/IA \quad S^{-1}R \otimes_R A \cong \varphi(S)^{-1}A.$$

(7) Let A and B be finite type k-algebras. The k-algebra $A \otimes_k B$ is to be interpreted as the functions on which space ? Now fix surjections $k[x_1, \ldots, x_n] \to A$ and $k[x_1, \ldots, x_n] \to B$. The k-algebra $A \otimes_{k[x_1,\ldots,x_n]} B$ is to be interpreted as the functions on which space ?
(8) Let R be a ring and M an R-module. Recall what is $\mathrm{Ann}(M)$ and show that if $I \leq \mathrm{Ann}(M)$ then M is naturally a R/I-module. Deduce for example that I/I^2 is an A/I module.

Where are we headed? We will introduce the theory of *schemes*. From the course on algebraic curves, you learned how to interpret *finite type k-algebras* as functions on closed subsets of $\mathbb{A}^n_k$, and saw that the study of such spaces was ultimately related to the algebras of their functions. With the theory of schemes, we will now interpret *any commutative ring as functions on some space.* For example, $\mathbb{Z}$ or any ring of integers can be interpreted as functions on some space, and also rings in finite characteristic. This unveils an all new range of geometric objects. One of the strength of the theory of schemes is that it is a general framework which captures not only the geometry of curves over $\mathbb{C}$ but also the geometry of objects that are more arithmetic in nature. The dictionary between algebra and geometry in the setting that you know and was recalled in a small amount in the preceding exercise will extend to the general setting of schemes.

Solution key. (5) The universal property of $R' \otimes_R R''$ is that a R-algebra map out of this to an R-algebra S is the same as a pair of maps of R-algebras $R' \to S$ and $R \to S$. It is therefore the *coproduct* of R and R' in the category of R-algebras.

(7) $A \otimes_k B$ is to be interpreted as function on the product of the associated closed subspaces of $\mathbb{A}^n_k$ and $A \otimes_{k[x_1,\dots,x_n]} B$ as functions on their intersection in $\mathbb{A}^n_k$.

□

The following exercises are about *sheaves*. Unless specifically mentioned, a sheaf means a *set*-valued sheaf. For a topological space X, $\mathrm{Op}(X)$ denotes the poset of opens of X.

Exercise 2. *$\mathcal{H}om$ sheaf*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf on X and $\mathcal{G}$ be sheaf on X. For any open $U \subset X$, denote by $\mathcal{F}_U$ the presheaf on U defined by $V \mapsto \mathcal{F}(V)$ for any $V \subset U$. Show that the presheaf $\mathcal{H}om : U \mapsto \mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$, where $\mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$ denotes the set of morphisms of (pre)sheaves on U, is a sheaf.

Solution key. We first expose a proof for sheaves of sets. Let (U_i) be an open cover of X.[1] Let $\varphi_i : \mathcal{F}_{U_i} \to \mathcal{G}_{U_i}$ be a collection of morphisms who agree on intersection. We show that it lifts uniquely to a morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

Let V be any open of X. Consider $s \in \mathcal{F}(V)$. Using that $\mathcal{G}$ is a sheaf, that morphisms agree on intersections, and that φ_i is a morphism of presheaves for all i, we get that $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$ lifts uniquely to an element of $\mathcal{G}(V)$ that we denote by $\varphi_V(s)$. We want to show that $(\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V))$ is a morphism of presheaves. To see that, note that if $V' \subset V$ and $s \in \mathcal{F}(V)$,

$$\varphi_{V'}(s_{V'})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_{i,V'\cap U_i}(s_{V'\cap U_i})$$

$$\overset{\varphi \text{ is a morphism of presheaves}}{=} \varphi_{i,V\cap U_i}(s_{V\cap U_i})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_V(s)_{|V'\cap U_i}$$

so both $\varphi_{V'}(s_{V'})$ and $\varphi_V(s)_{V'}$ restrict on $V' \cap U_i$ to the same element. As $\mathcal{G}$ is a sheaf, the desired equality follows. Note that for any $V \subset U_i$ we see by definition that $\varphi_V = \varphi_{i,V}$. This shows the existence of the lift.

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[1] This case will suffice; for a general open V we can apply the reasoning to $X = V$ and $\mathcal{F} = \mathcal{F}_{|V}$ and $\mathcal{G} = \mathcal{G}_{|V}$.

As for the unicity note that value on $s \in \mathcal{F}(V)$ of a lift φ' necessarily restricts to $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$. Therefore the uniqueness follows from the uniqueness in the sheaf property of $\mathcal{F}$.[2]

We answer now a question asked during TA sessions : can we do this with sheaves with value in an arbitrary category $\mathcal{C}$? The answer is yes and we will do some preliminary definitions. Note that in the above proof there is essentialy three steps: one commutative diagram to show the existence, one to show that this defines a natural transformation, and one argument for the unicity. The proof below is the same pattern.

Let $\mathcal{C}$ be a complete category. A sheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is a presheaf such that for any open U of X and open covering (U_i) of U, the following[3]

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

is an equalizer diagram. We denote by $\mathrm{Sh}_{\mathcal{C}}(X)$ the full subcategory of $\mathrm{Psh}_{\mathcal{C}}(X) = \mathrm{Fun}(\mathrm{Ouv}(X)^{op}, \mathcal{C})$ consisting of sheaves with values in $\mathcal{C}$.

Now we define the Set-valued presheaf

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathcal{C}}(U)}(\mathcal{F}_U, \mathcal{G}_U)$$

Now we want to show that this pre-sheaf is a sheaf, if we make the hypothesis that $\mathcal{G}$ is a sheaf. To show this, take $(U_i)_{i\in I}$ an open cover of $U \in \mathrm{Ouv}\, X$ and a collection of natural transformations

$$(\alpha^i : \mathcal{F}_{U_i} \to \mathcal{G}_{U_i})_{i\in I}$$

such that for all $i, j \in I$ and $W \subset U_{ij}$

$$(\alpha^i_W : \mathcal{F}(W) \to \mathcal{G}(W)) = (\alpha^j_W : \mathcal{F}(W) \to \mathcal{G}(W)). \tag{1}$$

We need to show that there is a unique natural transformation $\widehat{\alpha} : \mathcal{F}_U \to \mathcal{G}_U$ such that restricting this natural transformation to a U_i gives α_i.

Let $V \subset U$ be open. By the universal property of the product, let :

$$\beta_V : \mathcal{F}(V) \to \prod_{i\in I} \mathcal{G}(V \cap U_i)$$

induced by

$$\mathcal{F}(V) \to \mathcal{F}(V \cap U_i) \xrightarrow{\alpha^i_{V\cap U_i}} \mathcal{G}(V \cap U_i).$$

Now we want to consider $\widehat{\alpha}_V : \mathcal{F}(V) \to \mathcal{G}(V)$ the unique morphism who would be given the universal property of the following equalizer (because $\mathcal{G}$ is a sheaf) for the cover of V being $(V \cap U_i)_i$. Note that if $V \subset U_i$, by construction, we will have $\widehat{\alpha}_V = \alpha^i_V$.

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[2] If one now wants to show a similar statement for sheaves of abelian groups/rings/etc. one can now argue that to verify that a morphism of presheaves of sets is a morphism of presheaves of abelian groups/rings/etc. it suffices to check it at stalks/locally, which will hold because by construction it will already hold locally.

[3] with the two maps being on component (i, j) once $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_i) \to \mathcal{F}(U_{ij})$ and $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_j) \to \mathcal{F}(U_{ij})$ the other time

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[d, dashed, "\widehat{\alpha}_V"'] \arrow[dr, "\beta_V"] & & \\
\mathcal{G}(V) \arrow[r] & \prod_{i\in I}\mathcal{G}(V\cap U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j}\mathcal{G}(V\cap U_{ij})
\end{tikzcd}$$

To see that this works, we need to show that β_V commutes indeed in this diagram.
This holds, because of the commutative the diagram below, who commutes because $\mathcal{F}$ and $\mathcal{G}$ are functors, that α^i, α^j are natural transformations and that using (1) we have $\alpha^i_{V\cap U_{ij}} = \alpha^j_{V\cap U_{ij}}$.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V\cap U_i) \arrow[d] \arrow[dr, "\alpha^i_{V\cap U_i}"] & & \\
\mathcal{F}(V\cap U_j) \arrow[r] \arrow[dr, "\alpha^j_{V\cap U_j}"'] & \mathcal{F}(V\cap U_{ij}) \arrow[ddrr, "\alpha^i_{V\cap U_{ij}}=\alpha^j_{V\cap U_{ij}}"] & \mathcal{G}(V\cap U_i) \arrow[ddr, bend left] & \\
 & \mathcal{G}(V\cap U_j) \arrow[drr, bend right] & & \\
 & & & \mathcal{G}(V\cap U_{ij})
\end{tikzcd}$$

So $\widehat{\alpha}_V \colon \mathcal{F}(V) \to \mathcal{G}(V)$ is indeed well defined.
We claim that $(\widehat{\alpha}_V : \mathcal{F}(V) \to \mathcal{G}(V))_{V\subset U}$ is a natural transformation lifting the collection above.
We show that $\widehat{\alpha}$ is natural. This mean we have to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[d, "\widehat{\alpha}_{V'}"] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V')
\end{tikzcd}$$

By the universal property of the equalizer (using again that $\mathcal{G}$ is a sheaf), it amounts to prove the commutativity of,

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

So using the universal property of the product, we need only to verify that for every i :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[r] & \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

commutes. But this holds because we can insert commutating diagrams inside the diagram above in the following way :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[dr] \arrow[ddd, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] \arrow[dr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V' \cap U_i) \arrow[ddd] \\
& \mathcal{F}(V \cap U_i) \arrow[r] \arrow[dr, "\alpha^i_{V \cap U_i}"'] & \mathcal{F}(V' \cap U_i) \arrow[ddr, "\alpha^i_{V' \cap U_i}"] & \\
& & \mathcal{G}(V \cap U_i) \arrow[dr] & \\
\mathcal{G}(V) \arrow[r] \arrow[uurr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V' \cap U_i) \arrow[r] & \mathcal{G}(V' \cap U_i)
\end{tikzcd}$$

The intermediate diagrams commute because of the functoriality of $\mathcal{F}$ and $\mathcal{G}$, the naturality of α^i and the definition of $\widehat{\alpha}$.
The unicity of the lift is left to show. Suppose that $\widehat{\alpha'}$ is a lift. Then for any V, and $i \in I$ we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha'}_V"'] & \mathcal{F}(V \cap U_i) \arrow[d, "\alpha^i_{V \cap U_i}"] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V \cap U_i)
\end{tikzcd}$$

Therefore we see that by universal property of $\mathcal{G}(V)$ as an equalizer with respect to the sheaf property and the cover $(U_i \cap V)_i$ of V that $\widehat{\alpha'}_V = \widehat{\alpha}_V$.
□

Exercise 3. *Constant sheaves*
Consider the sheafification of the constant presheaf of $\mathbb{Q}$-vector spaces on the real line $\mathbb{R}$ defined by

$$U \in \mathrm{Op}(\mathbb{R}) \mapsto \mathbb{Q}.$$

We denote by this sheafification $\underline{\mathbb{Q}}$. Compute the value of $\underline{\mathbb{Q}}$ on any open subset of the real line. When the dimension of the $\mathbb{Q}$-vector space $\underline{\mathbb{Q}}(U)$ is finite ? In this case, what is this dimension ?

Solution key. Let S be a set and X a topological space. In what follows we prove that on a connected open subspace U the canonical map $S \to \underline{S}(U)$ is a bijection. We use the following description

$$\underline{S}(U) = \{(s_x) \in \prod_{x \in U} S \mid \forall x \in X \quad \exists U \ni x \quad \forall y, y' \in U \quad s_y = s_{y'}\}$$

and the natural map $S \to \underline{S}(U)$ being the diagonal. Let $(t_x) \in \underline{S}(U)$. Fix $y \in U$ (connected implies non empty). Now note that

$$V_1 = \{x \in U \mid t_x = t_y\} \quad V_2 = \{x \in U \mid t_x \neq t_y\}$$

form a disjoint decomposition of U into open subspaces. As U is connected and $y \in V_1$ we get $V_2 = \emptyset$ and the claim follows.

Now, as any subset U of the real line is a disjoint union of connected open subsets (which is also true for any locally connected space), we get that $\underline{\mathbb{Q}}(U) = \prod_{\pi_0(U)} \mathbb{Q}$ using the sheaf property. This vector space is finite dimensional when U has finitely many connected components and the dimension is then equal to $\pi_0(U)$.

☐

Exercise 4. *Sheaves and sections*

(1) Let X and Y be topological spaces, and $f : Y \to X$ a continuous map. Show that the following

$$\mathcal{F}_f(U) = \{s : U \to Y \text{ continuous} \mid f \circ s = \mathrm{id}_U\}$$

defines a sheaf on X. We call it the sheaf of section of f.

(2) Let $\mathcal{F}$ be a sheaf on a topological space X. Define the topological space

$$|\mathcal{F}| = \bigsqcup_{x \in X} \mathcal{F}_x$$

as a set with the finest topology such that for any $U \subset X$ open and $s \in \mathcal{F}(U)$ the section $x \mapsto s_x$ of the canonical map is continuous. Show that $|\mathcal{F}| \to X$ is a local homeomorphism and that the sheaf of section of this map is isomorphic to $\mathcal{F}$.

Solution key. (2) Everything in what follows works for a presheaf. Note first of all that any $s \in \mathcal{F}(V)$ the map $\widehat{s}\colon V \to |\mathcal{F}|$ defined by $x \mapsto s_x$ is a section of $p : |\mathcal{F}| \to X$. Note also that

$$\widehat{s}(V) = \{s_x \mid x \in V\}$$

is open. Indeed, we need to show by definition of the topology that for any V' open and $t \in \mathcal{F}(V')$

$$\widehat{t}^{-1}(\widehat{s}(V)) = \{x \in V \cap V' \mid s_x = t_x\}$$

is open. This follows from the following lemma about directed colimits.[4]

Lemma. *Let (A_i) be a directed system of sets and $\varinjlim_i A_i$ the colimit. If $a_i \in A_i$ and $a_j \in A_j$ coincide in the colimit, then there exists k with $i \to k$ and $j \to k$ with the image of a_i and a_j being the same in A_k.*

Proof. One checks that the colimit is given by the quotient of $\bigsqcup_i A_i$ by the relation $(a_i \in A_i) \sim (a_j \in A_j)$ if and only if there exist $i \to k$ and $j \to k$ with a_i and a_k identified in A_k. Once this understood, the lemma follows. ☐

Now, it follows that $p : |\mathcal{F}| \to X$ is continuous. Indeed for an open set U of X we have

$$p^{-1}(U) = \bigsqcup_{(s,V), s \in \mathcal{F}(V)} \widehat{s}(V).$$

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[4] As forgetful functors to sets from abelian groups or rings commute with directed colimits, this lemma also applies to directed colimits of abelian groups, rings.

Also, we see that for any open V and $s \in \mathcal{F}(V)$ we have $p_{|\widehat{s}(V)}\widehat{s} = \mathrm{id}_V$ and $\widehat{s}p_{|\widehat{s}(V)} = \mathrm{id}_{\widehat{s}(V)}$. Therefore p is a local homeomorphism because $p()$.

Remark. We have a natural isomorphism between $\mathcal{F}_p \to \mathcal{F}^+$. (Here $\mathcal{F}_p$ denotes the sheaf of sections of $p : |\mathcal{F}| \to X$.)

□

Remark. One can promote the construction $\mathcal{F} \mapsto |\mathcal{F}|$ to an equivalence of categories between $\mathrm{Sh}(X)$ and $\text{Ét}(X)$ the category of local homeomorphisms over X. For more details, see for example *Manifolds, sheaves, and cohomology* by Wedhorn.

Exercise 5. *Sheafification*
Let X be a topological space and $\mathcal{F}$ a presheaf on X. Show that the natural map $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks.
Find examples of topological spaces X and presheaves $\mathcal{F}$ on X such that

(1) The natural map $\mathcal{F} \to \mathcal{F}^+$ is not injective/resp. not surjective on some non empty open set.
(2) An abelian group valued presheaf with $\mathcal{F} \neq 0$ but $\mathcal{F}^+ = 0$.

Solution key. To show that $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks, we proceed as follows. Note that for any open $U \ni x$ the following projection map

$$\mathcal{F}^+(U) \subset \prod_{x \in U} \mathcal{F}_x \to \mathcal{F}_x$$

will pass to the colimit $(\mathcal{F}^+)_x \to \mathcal{F}_x$. One immediately checks that this is an inverse to the induced map at stalks from $\mathcal{F} \to \mathcal{F}^+$.
For (2) and "not injective" we can take the presheaf on $\mathbb{R}$ with value $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ and 0 for any other open.
For "not surjective", take $\mathbb{R}$ and the sheafification of any non-zero abelian group. See "constant" sheaf exercise 3.

□

Exercise 6. *Some sheaves on the circle*
We are using the notation $\mathcal{F}_f$ from exercise 4.

(1) Consider the map $e : [0, \frac{3}{2}] \to S^1$ defined by $t \mapsto \exp(2\pi i t)$. Compute all stalks of $\mathcal{F}_e$.
(2) Let $\mathcal{O}$ be the presheaf on S^1 defined for $U \in \mathrm{Op}(S^1)$ by

$$\mathcal{O}(U) = \{U \to \mathbb{R} \text{ continuous}\}$$

Show that $\mathcal{O}$ is a sheaf. Note that $\mathcal{O}$ is a sheaf of $\mathbb{R}$-algebras by acting pointwise. Show that for every $z \in S^1$, $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$.

Consider now the quotient M of $[0,1] \times \mathbb{R}$ by identifying $(0,t)$ with $(1,-t)$. Consider the map $\pi\colon M \to S^1$ defined by $\pi([x,t]) = \exp(2\pi i x)$. We also take the notation $\mathcal{F}_\pi = \mathcal{L}$.

(3) Show that for every $U \in \mathrm{Op}(S^1)$, $\mathcal{L}(U)$ is an $\mathcal{O}(U)$-module by $\mathcal{O}(U)$ acting on the second component.
(4) Show that for every open set $U \subset S^1$ with at least one point missing there is an isomorphism of sheaves $\mathcal{O}_U \cong \mathcal{L}_U$ which respects the module structure on evaluation on each open subset.
(5) Show that for every $s \in \mathcal{L}(S^1)$ there exist a $z \in S^1$ such that $s(z) = [z, 0]$.
(6) Deduce that there is *no* isomorphism $\mathcal{O} \cong \mathcal{L}$ of sheaves respecting the module structure on each open subset.

Solution key. (1) Note that $e : [0, \frac{3}{2}] \to S^1$ is a local homeomorphism. We claim that the natural evaluation map

$$(\mathcal{F}_e)_z \xrightarrow{\mathrm{ev}_z} e^{-1}(z)$$

is a bijection.[5] Let $x \in e^{-1}(z)$. Let $U \ni Z$ such that $e_{|U}$ is an homeomorphism. Then $e_{|U}^{-1}(z) = x$. This shows surjectivity. If s, t are sections on say $V \ni Z$ and $V' \ni z$ which have the same value on z, say x, then take an open $U \ni x$ such that $e_{|U}$ is an homoemoprhism and $e(U) \subset V \cap V'$. Then $s_{|e(U)}$ and $t_{|e(U)}$ are both the unique inverse to $e_{|U}$. This shows the injectivity.

(2) We show that $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra. We claim that the ideal

$$\{f \in \mathcal{O}_z \mid f(z) = 0\}$$

is the unique maximal ideal. To this end, it suffices to show that the complement consits of the invertible elements. If $f(z) \neq 0$, then there exists a neighbourghood of z where f never vanishes. Therefore $\frac{1}{f}$ is a well defined multiplicative inverse in the stalk.

Some setup and notations for the rest of the exercise.

(a) To avoid confusion, we write the complex number $e(0) = e(1) = 1 \in S^1$ by u.

(b) Denote by $e\colon [0,1] \to S^1$ the quotient map given by $\exp(2\pi i -)$.

(c) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon (0,1) \times \mathbb{R} \to \pi^{-1}(S^1 \setminus u).$$

(d) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [0, \frac{1}{2}] \times \mathbb{R} \to \pi^{-1}(S^1_{\geq 0}),$$

where $S^1_{\geq 0}$ denotes the points of the circle with imaginary part positive or zero.

(e) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [\frac{1}{2}, 1] \times \mathbb{R} \to \pi^{-1}(S^1_{\leq 0}),$$

where $S^1_{\leq 0}$ denotes the points of the circle with imaginary part negative or zero.

Let $s \in \mathcal{L}(U)$ be a section. We define a continuous map $\alpha_s\colon e^{-1}(U) \to \mathbb{R}$ such that

$$s(e(t)) = [e(t), \alpha_s(t)].$$

For $t \neq 0, 1$, we define $\alpha_s(t)$ to be the second component of $p^{-1}(s(e(t)))$, by (c) above. When $t = 0$ and $t = 1$, we extend by continuity and the same method using the points (d) and (e) respectively. Note that

$$\alpha_s(0) = -\alpha_s(1)$$

because $s(u) = [0, \alpha_s(0)] = [1, \alpha_s(1)]$.

(3) We define a module structure. We explain how to define the multiplication by scalars, the others operations being defined similarly. Let U be any open of M. Let $f \in \mathcal{O}(U)$ and $s \in \mathcal{L}(U)$. We define $f \cdot s$ as follows. We pass to the quotient map $e\colon [0,1] \to S^1$, the following continuous map $[0,1] \to M$

$$t \mapsto [t, f(e(t))\alpha_s(t))].$$

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[5] Note that the following argument holds true for any local homeomorphism $e : X \to Y$.

To show that it passes to the quotient we have to show that it agrees on $t = 0$ and $t = 1$. But as

$$f(u)\alpha_s(0) = f(u)(-\alpha_s(1)) = -f(u)\alpha_s(1),$$

this follows from the quotient relation of the Möbius band.

The zero element is the section $s_0 : S^1 \to M$, $s_0(e(t)) = [t, 0]$.

One continue similarly to define the rest of the structure. The key is that the "gluing of the quotient" $(-1)\colon \mathbb{R} \to \mathbb{R}$ is an automorphism of $\mathbb{R}$-modules so that we can "lift" calculations to pointwise calculations in $[0, 1] \times \mathbb{R}$. That's why we put the emphasis on that in the above calculation.

(4) For any section $s \in \mathcal{L}(U)$ we have the unique map

$$\mathcal{O}_{|U} \to \mathcal{L}_{|U}$$

that respects the module structure on each open subset of U and sends 1 to s. We claim that if s vanishes nowhere, then this map is an isomorphism. To prove that, we suppose that s vanishes nowhere, and construct an homeomorphism over U

$$\psi_s \colon \pi^{-1}(U) \to U \times \mathbb{R}$$

defined by $[t, \lambda] \mapsto (e(t), \frac{\lambda}{\alpha_s(t)})$. This is well defined by non-vanishing. The inverse is given by $(z, \lambda) \mapsto (\lambda \cdot s)(z)$, where $\cdot$ designates the module structure defined above. Now

$$pr_2\psi_s(-)\colon \mathcal{L}_{|U} \to \mathcal{O}_{|U}$$

gives an inverse to the above map.

We are now left to prove that on any open subset missing a point U, there exist a non-vanishing section. But whenever a point is missing, say $e(t_0) \notin U$ for some $t_0 \in [0, 1)$ then we can define the section $U \to M$ by

$$e(t) \mapsto \begin{cases} [t, 1] & t < t_0 \\ [t, -1] & t > t_0 \end{cases}$$

which vanishes nowhere.

(5) Let $s \in \mathcal{L}(S^1)$. By the intermediate value theorem $\alpha_s\colon [0, 1] \to \mathbb{R}$ necessarily vanishes because $\alpha_s(0) = -\alpha_s(1)$.

(6) Note that a section $s \in \mathcal{L}(U)$ vanishes at $z = e(t)$ in the sense that $s(z) = [t, 0]$ if and only if $s_z \in \mathfrak{m}_z\mathcal{L}_z$. Note that $1 \in \mathcal{O}(S^1)$ vanishes on no point. By contradiction, the image of 1 by an isomorphism $\mathcal{O} \cong \mathcal{L}$ would not vanish at any stalk, in contradiction with the previous point.

□

Exercise 7. *Skyscraper sheaves*

For any set S and $x \in \mathbb{R}$ show that the following defines a sheaf, for $U \in \mathrm{Op}(\mathbb{R})$

$$x_*S(U) = \begin{cases} S \text{ if } x \in U \\ \mathbb{1} \text{ if } x \notin U, \end{cases}$$

where $\mathbb{1}$ is the set with one element. We call this sheaf the *skyscraper sheaf of* S *at* x. Compute every stalk of x_*S. Understand and draw the topological space $|x_*S|$ (see exercise 4). Do you understand the name *skyscraper* now ?

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EPFL, fall semester 2024
AG II - Schemes and sheaves

Solutions – week 2

Exercise 1. *Sheaves of abelian groups.* Let X be a topological space. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphisms of sheaves of abelian groups.

1. Let $\ker(\varphi)$ and $\operatorname{im}(\varphi)$ be respectively the kernel sheaf and the image sheaf.[1] Show that for every $x \in X$, one can define natural maps which are isomorphisms
$$\ker(\varphi)_x \to \ker(\varphi_x) \text{ and } \operatorname{im}(\varphi)_x \to \operatorname{im}(\varphi_x).$$
2. Show that φ is an injective morphism of sheaves (resp. surjective morphism of sheaves) if and only if for every $x \in X$ the morphism of abelian groups $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).
3. Show that φ is a surjective morphism of sheaves if and only if for every U open and $s \in \mathcal{G}(U)$, there exists an open cover $U = \bigcup U_i$ and sections $t_i \in \mathcal{F}(U_i)$ with $\varphi(t_i) = s_{U_i}$.
4. Show that the natural map $\operatorname{im}(\varphi) \to \mathcal{G}$ is injective.
5. Show that φ is an isomorphism if and only if it is an injective morphism of sheaves and a surjective morphism of sheaves.
6. Let $f = X \to *$ be the unique morphism to the point. Show that $f_* = \Gamma(X, -) : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{Ab}$ is left-exact. Give an example to show that f_* is not right-exact in general.

Exercise 2. *Gluing sheaves.* Let X be a topological space and $\bigcup U_i = X$ an open cover of X. Let $(\mathcal{F}_i \in \mathrm{Sh}(U_i), \varphi_{ij})$ be a collection of sheaves on $\mathrm{Sh}(U_i)$ together with isomorphisms
$$\varphi_{ij} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$$
in $\mathrm{Sh}(U_{ij})$ satisfying for each i that $\mathrm{id} = \varphi_{ii}$ and for each i, j, k the following *cocycle condition* $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.
Show that there exists a unique[2] sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ with maps $\psi_i : \mathcal{F}_{|U_i} \to \mathcal{F}_i$ with the following universal property: for all sheaves $\mathcal{G} \in \mathrm{Sh}(X)$ we have a a bijection
$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \cong \left\{ (\mathcal{G}_{|U_i} \xrightarrow{f_i} \mathcal{F}_i) \in \prod_i \mathrm{Hom}(\mathcal{G}_{|U_i}, \mathcal{F}_i) \mid \text{s.t. for all } i, j : \varphi_{ij} f_i = f_j \right\}$$
given by $f \mapsto \psi_i f_{|U_i}$.
Show furthermore that ψ_i are isomorphisms.

---

[1] The kernel sheaf is the kernel presheaf but the image sheaf is the *sheafification* of the image presheaf.

[2] up to isomorphism.

Solution key. Let $\iota_i : U_i \to X$ for the inclusion of the open set.
Some quick remarks : using cocycle condition, we get $\varphi_{ii} = \varphi_{ii} \circ \varphi_{ii}$. By hypothesis φ_{ii} are isomorphisms so we get : $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$. Then using the cocycle condition : $\mathrm{id}_{\mathcal{F}_i} = \varphi_{ji} \circ \varphi_{ij}$ and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$.
We define [3] $\mathcal{F}$ on an open set U by :

$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}_i(U \cap U_i) \mid \forall (i,j) \quad s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}})\}$$

as *sub-(pre)sheaf of the product sheaf* $\prod_i \iota_{i*}\mathcal{F}_i$ If $V \subset U$ note that the restriction $(s_i) \mapsto (s_{i|_V})$ is well defined because : $s_{j|_{U_{ij}\cap V}} = \varphi_{ij}(s_{i|_{U_{ij}}})_{|V} = \varphi_{ij}(s_{i|_{U_{ij}\cap V}})$, using the fact that φ_{ij} is a morphism of sheaves.

- We show that $\mathcal{F}$ is indeed a sheaf. Let $V = \cup_\alpha V_\alpha$ an open cover. Let $((s_i^\alpha)_i)_\alpha$ be a collection of elements lying in $\mathcal{F}(V_\alpha)$, such that we have $s_i^\alpha{}_{|V_\alpha} = s_i^\beta{}_{|V_\beta}$ for any α, β. Using the sheaf property on the product sheaf (which follows directly from the sheaf property of each factor), we get a *unique* element $(s_i) \in \prod_i \iota_{i*}\mathcal{F}_i(V)$ lifting the collection. We show that this unique element lies in fact in $\mathcal{F}(V)$. We need to show that for any i, j we have $s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}})$. But when we restrict both sides of the desired equality on V_α, the equality holds because $(s_i^\alpha)_i$ lies in $\mathcal{F}(V_\alpha)$. So using again the uniqueness in the sheaf property of the product sheaf, we get what we want.
- To show that $\mathcal{F}$ is unique up to isomorphism in $\mathrm{Sh}(X)$ we spell out an universal property that it verifies. We write $(\mathcal{F} \xrightarrow{p_i} \iota_{i*}\mathcal{F}_i)_i$ the collection of sheaf morphisms induced by the projections from the product. We claim that $\mathcal{F}$ satisfies the following universal property :

  For all $\mathcal{G} \in \mathrm{Sh}(X)$ and collections $(\mathcal{G} \xrightarrow{f_i} \iota_{i*}\mathcal{F}_i)_i$ of sheaf morphisms
  such that :
  for all U open and $\forall t \in \mathcal{G}(U)$, we have for all i, j :
  $f_j(t)_{|Uij} = \varphi_{ij}(f_i(t)_{|Uij})$
  there is a unique sheaf morphism $\mathcal{G} \xrightarrow{f} \mathcal{F}$ such that for all i,
  $p_i f = f_i$.

  This is indeed the case : if we take a collection $(\mathcal{G} \xrightarrow{f_i} \iota_{i*}\mathcal{F}_i)_i$, we get a map f from $\mathcal{G}$ to the product $\prod_i \iota_{i*}\mathcal{F}_i$ by the universal property of the product in $\mathrm{Sh}(X)$. But the condition $f_j(t)_{|Uij} = \varphi_{ij}(f_i(t)_{|Uij})$ for all i, j says exactly that in fact f factors into $\mathcal{F}$.
- Now we show that $\varphi_k : \mathcal{F}_{U_k} \to \mathcal{F}_k$ induced by the projection is an isomorphism of sheaves for all k. To show surjectivity we will use crucially the cocycle condition.

---

[3] One should question the coherence of this definition : let $(s_i) \in \mathcal{F}(U)$. Then

$$s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}}) = \varphi_{ij}(\varphi_{ji}(s_{j|_{U_{ij}}})) = s_{j|_{U_{ij}}} \tag{1}$$

using the property for (i,j) and (j,i) and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$. So (1) highlight why the fact that φ_{ij} and φ_{ji} are inverses to each other is important in this gluing process.

(1) Surjectivity. Let $V \subset U_k$ open. Let $s_k \in \mathcal{F}_k(V)$. We want to construct an element $(s_i) \in \mathcal{F}(V) \subset \prod_i \mathcal{F}_i(V \cap U_i)$ such that its k-th component is s_k.
For each i, we define s_i using the cover $V \cap U_i = \cup_j V \cap U_{ij}$, and the collection $(\varphi_{ki}(s_{k|_{Uij}}))$ of elements in $\mathcal{F}_i(V \cap U_{ij})$. It verifies the intersection property because φ_{ij} is a morphism of sheaves. So s_i is defined by $s_{i|_{U_{ij}}} = \varphi_{ki}(s_{k|_{Uij}})$. Note that if $i = k$ the element defined in this way is s_k, because φ_{kk} is the identity.
Now we claim that the collection (s_i) that we just defined is indeed in $\mathcal{F}(V)$. To show this, we need to show that for any i, j, we have : $s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}})$. But :

$$s_{j|_{U_{ij}}} = \varphi_{kj}(s_{k|_{Uij}}) = \varphi_{ij} \circ \varphi_{ki}(s_{k|_{Uij}}) = \varphi_{ij}(\varphi_{ki}(s_{k|_{Uij}})) = \varphi_{ij}(s_{i|_{U_{ij}}})$$

Using the definition of s_i's and the cocycle condition.

(2) Injectivity. Let (s_i) and (s'_i) be two elements in $\mathcal{F}(V)$ such that their k-th component is $s_k = s'_k$. Now one gets for any i :

$$s_i = \varphi_{ik}(s_k) = \varphi_{ik}(s'_k) = s'_i$$

thus proving the injectivity.

Remark. One can interpret the result of the previous exercise as sying that the presheaf with values in categories

$$\mathrm{Sh}\colon \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Cat}$$

is a sheaf in a suitable sense.

□

Remark. Can you see how the last exercise resembles the following statement: "$U \mapsto \mathrm{Sh}(U)$ is a sheaf"?

Exercise 3. *Inverse image.* Let $f\colon X \to Y$. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. We define the presheaf on X

$$f^\sharp\mathcal{F}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

(1) Show that if $f\colon * \to X$ is a point $x \in X$ then $f^\sharp\mathcal{F} = \mathcal{F}_x$.
(2) Show that if $y = f(x)$ then there is a natural isomorphism

$$(f^\sharp\mathcal{F})_x \to \mathcal{F}_y.$$

(3) Show that if f is an open immersion, then $f^\sharp$ is a sheaf.
(4) Find an example of map of topological spaces $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y such that $f^\sharp\mathcal{F}$ is *not* a sheaf.
(5) Let $f^{-1}\mathcal{F}$ be the sheafification of $f^\sharp\mathcal{F}$. We call this sheaf the *inverse image* of $\mathcal{F}$. Show that the $f^{-1} \dashv f_*$[4] meaning that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G}).$$

---

[4] We say that f^{-1} is *left adjoint to* f_*

Exercise 4. *Localization* Let R be a ring. Let S be a multiplicative subset.

1. Describe the points of $\mathrm{Spec}(S^{-1}R)$. If $\mathfrak{p} \in \mathrm{Spec}(R)$ show that $\mathrm{Spec}(R_\mathfrak{p})$ is the intersection of all opens containing $\mathfrak{p}$.
2. Let M be an R-module and $I \supseteq R$ an ideal. Show that there is an isomorphism
$$S^{-1}(M/I) \cong (S^{-1}M)/(IS^{-1}M).$$
3. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $I \leq R$ and ideal. When
$$(R/I)_\mathfrak{p} = 0 \quad ?$$
Can you interpret this geometrically?
4. Let R be integral. Identify the image of the injective map $S^{-1}R \to \mathrm{Frac}(R)$.
5. Let $R = \mathbb{Z}[x]$. Describe the localization at the maximal ideal (p, x).

Exercise 5. *Affine schemes are quasi-compact.* Let R be a ring. Show that $\mathrm{Spec}(R)$ is quasi-compact.[5] Deduce that the underlying topological space of any (affine) scheme has a basis of quasi-compact open subsets.

Solution key. Let R be a ring. Let (a_i) be a collection of elements such that
$$\mathrm{Spec}(R) = \bigcup_i D(a_i).$$
This means that $1 \in (a_i)$. Therefore there exists $a_1, \dots, a_n$ and $b_1, \dots, b_n \in R$ such that
$$1 = \sum_{j=1}^{n} b_j a_j$$
for some n. Therefore
$$\mathrm{Spec}(R) = \bigcup_{j=1}^{n} D(a_j).$$
□

Exercise 6. *Connected affine schemes.* We say that a ring R is *connected* if for all $a, b \in R$ if
$$a + b = 1 \text{ and } ab = 0$$
then exactly one of the two elements is non-zero.

1. Show that it is equivalent to the fact there is exactly two idempotents (namely 0 and 1) in the ring R.
2. Show that R is connected if and only if $\mathrm{Spec}(R)$ is connected.

Exercise 7. *Stalks, morphisms and cotangent spaces*

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[5] A topological space X is *quasi-compact* if every open cover of X can be refined to a finite cover.

(1) Let $X \to Y$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf on X. Let $x \in X$ and $y = f(x)$. Show that there is a natural map
$$(f_*\mathcal{F})_y \to \mathcal{F}_x.$$
Remark. This is used to define the *induced map on local rings* of a map of locally ringed spaces. Namely if $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is one, with $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, the induced map on local rings is for $x = f(y)$,
$$\mathcal{O}_{Y,y} \xrightarrow{f^\sharp_y} (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}.$$

(2) Let R be an integral domain. Consider $\varphi\colon R[x,y] \to R[x,y]$ defined by $x \mapsto xy$ and $y \mapsto y$. Consider
$$f\colon \operatorname{Spec}(R[x,y]) \to \operatorname{Spec}(R[x,y])$$
the induced map on associated affine schemes.[6] Show that for all $\lambda \in R$ we have $f((x-\lambda, y)) = (x, y)$.

(3) Let now $R = k$ a field. With point (1) and the remark there is induced map on local rings
$$k[x,y]_{(x,y)} \to k[x,y]_{(x-\lambda,y)}.$$
We write $\mathfrak{m}_{(0,0)} := \mathfrak{m}_{(x,y)}$ and $\mathfrak{m}_{(\lambda,0)} := \mathfrak{m}_{(x-\lambda,y)}$ for the maximal ideals of these local rings. Understand the induced k-linear map
$$\mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)} \to \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}^2_{(\lambda,0)}.$$
This mean the following: find a k-basis of these vector spaces and describe the matrix of the map in term of your chosen basis.
Remark. We will later see that these vector spaces are the *cotangent spaces* at $(0,0)$ and $(\lambda,0)$ respectively and that the map that you studied is *the precompostion by the differential of f at these points.*

Solution key. This exercise was a previous hand in exercise, and solutions are credited to past students of the course.

(2)(Alissa) Let R be an integral domain. Consider $\phi : R[x,y] \to R[x,y]$ a ring homomorphism such that $x \mapsto xy$ and $y \mapsto y$. Consider now the map $f : \operatorname{Spec}(R[x,y]) \to \operatorname{Spec}(R[x,y])$ induced by the map ϕ. We show that for every $\lambda \in R$ we have that $f((x-\lambda, y)) = \phi^{-1}(x-\lambda, y) = (x,y)$. To prove this point, consider the following commutative diagram

$$\begin{tikzcd}
R[x,y] \arrow[rr, "\phi"] \arrow[dr, "\mathrm{ev}_{(0,0)}"'] & & R[x,y] \arrow[dl, "\mathrm{ev}_{(\lambda,0)}"] \\
& R &
\end{tikzcd}$$

We have that $\mathrm{ev}_{(\lambda,0)} \circ \phi = \mathrm{ev}_{(0,0)}$. Hence we have the following series of equalities
$$\begin{aligned}(x,y) &= \ker(\mathrm{ev}_{(0,0)}) = \mathrm{ev}^{-1}_{(0,0)}(0) = (\mathrm{ev}_{(\lambda,0)} \circ \phi)^{-1}(0) \\ &= \phi^{-1}(\mathrm{ev}^{-1}_{(\lambda,0)}(0)) = \phi^{-1}(x-\lambda, y) = f(x-\lambda, y)\end{aligned}$$
This proves our point.

---

[6] Recall that the induced map on Spec is given by the *preimage* φ^{-1}

(3)(Maxence) Now let $R = k$ be a field. We have a local homomorphism of local rings

$$f^{\sharp}_{(x,y)} : \mathcal{O}_{\mathrm{Spec}(k[x,y]),(x,y)} \to \mathcal{O}_{\mathrm{Spec}(k[x,y]),(x-\lambda,y)}.$$

But we know that $\mathcal{O}_{\mathrm{Spec}(k[x,y]),\mathfrak{p}} \cong k[x,y]_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(k[x,y])$. Thus we can see $f^{\sharp}_{(x,y)}$ as a local homomorphism of local rings $k[x,y]_{(x,y)} \to k[x,y]_{(x-\lambda,y)}$. Set $\mathfrak{m}_{(0,0)}$ and $\mathfrak{m}_{(\lambda,0)}$ to be the maximal ideal of respectively $k[x,y]_{(x,y)}$ and $k[x,y]_{(x-\lambda,y)}$.

We want to understand the k-linear map $\mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)} \to \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}^2_{(\lambda,0)}$. For any maximal ideal $\mathfrak{m}$ of $k[x,y]$, we have the following isomorphism of k-vector spaces ($k[x,y]/\mathfrak{m}$-vector spaces) :

$$\mathfrak{m}_{\mathfrak{m}}/\mathfrak{m}^2_{\mathfrak{m}} \cong \mathfrak{m}/\mathfrak{m}^2$$

where $\mathfrak{m}_{\mathfrak{m}}$ is the maximal ideal of $k[x,y]_{\mathfrak{m}}$.

Furthermore, it is easy to see that $\{\overline{x}, \overline{y}\}$ is a k-basis of $(x,y)/(x^2, xy, y^2)$ and $\{\overline{x-\lambda}, \overline{y}\}$ is k-basis of $(x-\lambda, y)/((x-\lambda)^2, (x-\lambda)y, y^2)$ since they are k-linear independent elements in their respective quotient. Since the induced linear map is just defined by applying φ, we get that $\varphi(\overline{x}) = \overline{xy} = \lambda\overline{y}$ and $\varphi(\overline{y}) = \overline{y}$ by definition of elements in the quotient $(x-\lambda, y)/((x-\lambda)^2, (x-\lambda)y, y^2)$.

That is, by taking bases as above, the linear map that we are looking for can be describe as the following matrix

$$\begin{pmatrix} 0 & 0 \\ \lambda & 1 \end{pmatrix}.$$

□

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AG II - Schemes and sheaves

Solutions – week 3

Exercise 1. *Nilradical.* Let R be a ring. Denote by

$$\mathrm{nil}(R) := \{f \in R \mid f \text{ is nilpotent}\}.$$

(1) Show that

$$\mathrm{nil}(R) = \bigcap_{\mathfrak{p}\in \mathrm{Spec}(R)} \mathfrak{p}.$$

(2) Show that for an ideal $I \subset R$, we have $V(I) = \mathrm{Spec}(R)$ if and only if every element of I is nilpotent, meaning $I \subset \mathrm{nil}(R)$.

Exercise 2. Spec *is an adjoint.* Let $(X, \mathcal{O}_X)$ be a scheme and A a ring. Show that the induced map on global sections

$$\mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X))$$

is bijective. This implies that

$$\mathrm{Spec}\colon \mathrm{Ring}^{op} \to \mathrm{Sch}$$

is a right adjoint. In particular colimits of rings are sent to limits of schemes.

Solution key. It is straightforward to check the naturality of the map in X and A. We then just need to construct an inverse map to

$$\mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X)).$$

We proceed in three steps. (See the document *Gluing arguments* for more precision on some points.)

(1) If X is affine, this is bijective because this is the statement of the anti-equivalence of categories between affine schemes and rings.

(2) Suppose then that X can be covered by affines

$$X = \bigcup_i U_i$$

such that their intersection is affine. We construct an inverse map. Let $\varphi\colon A \to \mathcal{O}_X(X)$ be a ring map. Denote by $\varphi_i\colon A \to \mathcal{O}_X(U_i)$ the composition of φ with the restriction. Using the anti-equivalence between rings and affine schemes (1), we get that φ_i correspond uniquely to a map of schemes $f_i\colon U_i \to \mathrm{Spec}(A)$. We want to show that f_i and f_j coincide on $U_i \cap U_j$. As this intersection is affine by hypothesis we get that the restriction of f_i and f_j is the unique map of affine schemes which correspond to the map $\varphi_{ij}\colon A \to \mathcal{O}_X(U_{ij})$ which is φ composed by the restriction. Therefore we get a map of schemes $f\colon X \to \mathrm{Spec}(A)$. We check that it is the desired inverse. If $\varphi\colon A \to \mathcal{O}_X(X)$ is a ring map, the map on global sections induced by the above constructed f is φ by construction of the glued map. The other way around, if f is a map $X \to \operatorname{Spec}(A)$, we see by restricting to U_i that f is necessarily given by gluing of the maps induced by the above construction.

(3) We now consider X to be an arbitrary scheme. We want to construct an inverse map. We proceed exactly as above. The only difference is in the step when we want to compare f_i and f_j on $U_i \cap U_j$, which is not necessarly affine. But $U_i \cap U_j$ is a scheme that can be covered with affine schemes such that their intersection is affine (see *Gluing arguments.*) Therefore we can use (2) to say that a map $U_i \cap U_j \to \operatorname{Spec}(A)$ is the same as a map of global sections $A \to \mathcal{O}_X(U_i \cap U_j)$. Therefore f_i and f_j are the same because they correspond to the the map $\varphi_{ij} \colon A \to \mathcal{O}_X(U_{ij})$ which is φ composed by the restriction as in the above case. Every other step goes similarly.

□

Remark. The above remains true if we replace Sch by the category of locally ringed spaces $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}}$. This characterizes Spec as the right adjoint of the global sections functor $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}} \to \mathrm{Ring}^{op}$. This formalize the saying that $\operatorname{Spec}(R)$ is the universal (locally ringed) space such that R is the ring of global functions on this space.

Exercise 3. *Reduced schemes.* A scheme $(X, \mathcal{O}_X)$ is *reduced* if for all opens U of X the ring $\mathcal{O}_X(U)$ is reduced.

(1) Show that a scheme $(X, \mathcal{O}_X)$ is reduced if and only if for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a reduced ring.
(2) Show that an affine scheme $\operatorname{Spec}(A)$ is reduced if and only if A is a reduced ring.

The *reduction* of a scheme X is a scheme X_{red} together with a map $\iota : X_{red} \to X$ with the property that for every map $Y \to X$ where Y is a reduced scheme, then Y factors uniquely to ι.

(3) Show that if $X = \operatorname{Spec}(A)$ then $\operatorname{Spec}(A/\operatorname{nil}(A)) \to \operatorname{Spec}(A)$ is the reduction of $\operatorname{Spec}(A)$.
(4) Show that the reduction of any scheme exists and that $\iota \colon X_{red} \to X$ is a homeomorphism.

Solution key. (1) Suppose that $(X, \mathcal{O}_X)$ is reduced. Take $s_x \in \mathcal{O}_{X,x}$ such that $s_x^n = 0$. First take an U where s_x lifts to a section $s \in \mathcal{O}_X(U)$. Then s^n is sent to 0 in $\mathcal{O}_{X,x}$. It implies that there is a smaller open V such that $s^n = 0$. But as $\mathcal{O}_X(V)$ is reduced, we deduce that $s = 0$ in $\mathcal{O}_X(V)$ proving that $s_x = 0$ as wanted.

For the other direction, take $f \in \mathcal{O}_X(U)$ nilpotent. Then every image in all stalks for all $x \in U$ are nilpotent implying that $f_x = 0$ for all $x \in U$ and then $f = 0$.

(2) If $\operatorname{Spec}(A)$ is reduced then taking global sections we deduce that A is reduced as a ring.

For the other way around, we prove the following:

Claim. *If S is any multiplicative subset of A and A is reduced, then $S^{-1}A$ is also reduced.*

Indeed, if $\frac{a^n}{s^n} = 0$, it means that there is some N and $s' \in S$ such that ${s'}^N a^n = 0$. But then, we see that $s'a$ is nilpotent, of order at most $M = \max\{N, n\}$. As A is reduced, $s'a = 0$ implying that a is mapped to zero in $S^{-1}A$.

Therefore for every prime $\mathfrak{p}$ of A, $A_{\mathfrak{p}}$ is reduced, showing that $\mathrm{Spec}(A)$ is reduced.

(3) We show that $\mathrm{Spec}(A_{red}) \to \mathrm{Spec}(A)$ is the reduction in the category of schemes. Let $Y \to \mathrm{Spec}(A)$ a map, where Y is a reduced scheme. By adjunction, this is the same as the data of a map $A \to \mathcal{O}_Y(Y)$. Because the target is reduced, this map factors uniquely to $A \to A_{red}$. By adjunction again, we get the unique desired map $Y \to \mathrm{Spec}(A_{red})$.

(4) Define a scheme X_{red} with the same underlying topological space, but with $\mathcal{O}_{X_{red}}$ being the sheafification of $U \to \mathcal{O}_X(U)_{red}$. Let (U_i) be a basis of X consisting only of affine open sub-schemes. For every open affine $U_i = \mathrm{Spec}(A_i)$ the presheaf define above is equal to $\mathcal{O}_{\mathrm{Spec}(A_{i,red})}$ on open affines of U_i. Therefore, because this presheaf already defines a sheaf on a basis of open subsets, this implies that the sheafification equals it on these affine opens (but not necessarily on other opens). Therefore we conclude that $\mathcal{O}_{\mathrm{Spec}(A_{i,red})}$ is equal to the sheafification of the presheaf defined above on $\mathrm{Spec}(A_i)$. It follows that X_{red} with the same toplogical space as X and the sheaf define above is a scheme.

Now for the universal property, if $f\colon Y \to X$ is a morphism with Y reduced, then topologically there is evidently a unique lift $Y \to X_{red}$. For the sheaf part consider the map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y.$$

Because $f_*\mathcal{O}_Y$ is reduced on every open this factors uniquely through the presheaf reduction and then to the sheafification $\mathcal{O}_{X,red}$ by universal property of the sheafification. This is what we wanted.

□

Exercise 4. *Residue fields and rational points.* Let $(X, \mathcal{O}_X)$ be a scheme, $x \in X$ and $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ the *residue field at x.*

(1) Let K be a field. Show that a map $\mathrm{Spec}(K) \to X$ with topological image x amounts to a field extension $k(x) \to K$.

(2) Let k be a field. Fix $X \to \mathrm{Spec}(k)$ a map for the rest of the exercise. Show that for all $x \in X$, $k(x)$ is naturally a field extension of k.

(3) We say that $x \in X$ is *k-rational* if the natural extension of last item $k \to k(x)$ is an isomorphism. Show that the set of k-rational points of X is identified with the set of maps $\mathrm{Spec}(k) \to X$ such that the composite $\mathrm{Spec}(k) \to X \to \mathrm{Spec}(k)$ is the identity.

(4) Let now $X = \mathrm{Spec}(k[x_1, \dots, x_n]/(f_1, \dots, f_m)) \to \mathrm{Spec}(k)^1$, where $f_1, \dots, f_m$ are polynomials. Show that the set of k-rational points of X is identified with the set of solutions in k^n of the system of polynomials $f_1, \dots, f_m$.

Exercise 5. *Exceptional functors (1).* Let X be a topological space. Let $j\colon U \to X$ be an open subset and $\iota\colon Z \to X$ its closed complement. We work with categories of sheaves of abelian groups on these spaces.

(1) Consider $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$. Compute every stalk of $\iota_*\mathcal{F}$.
(2) Show that ι_* is exact.
(3) Give an example to show that j_* is not exact.

Consider $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$. We define the *extension by zero* or *exceptional direct image* $j_!\mathcal{G}$ to be the sheafification of the presheaf defined by $V \mapsto \mathcal{G}(V)$ if $V \subset U$ and 0 otherwise.

(4) Show that for every sheaf $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$ there is a natural exact sequence

$$0 \to j_!j^{-1}\mathcal{H} \to \mathcal{H} \to \iota_*\iota^{-1}\mathcal{H} \to 0.$$

(5) Show that there is a natural bijection in $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ and $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(U)}(\mathcal{G}, j^{-1}\mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(j_!\mathcal{G}, \mathcal{H}).$$

This means that for an open immersion j, we have a sequence of adjoints $j_! \dashv j^{-1} \dashv j_*$.

Solution key. (1) If $x \in Z$ then we see that we have a natural isomorphism

$$(\iota_*\mathcal{F})_x \cong \mathcal{F}_x.$$

If $x \notin Z$ then as $\iota_*\mathcal{F}(X \setminus Z) = \mathcal{F}(\emptyset) = 0$ we see that $(\iota\mathcal{F})_x = 0$.

(2) To check the exactness of a sequence, we check it at stalks. Therefore the exactness of ι_* follows from the previous computation.

(3) Consider $U = \mathbb{C} \setminus 0 \to \mathbb{C}$. Consider the exponential sequence ($\mathcal{O}$ denotes sheaves of holomorphic functions and $\mathcal{O}^\times$ the sheaf of non-vanishing holomorphic functions)

$$1 \to \mathbb{Z} \to \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^\times \to 1.$$

We claim that $j_*\mathcal{O}_U \to j_*\mathcal{O}_U^\times$ is not surjective. By contradiction, if it is, it would be surjective at the stalk at zero

$$(j_*\mathcal{O}_U)_0 \to (j_*\mathcal{O}_U^\times)_0.$$

In particular the germ of the inclusion map $g\colon U \to \mathbb{C} \setminus 0$ would be attained by some element. This means that there exists $V \subset U$ with $f \in \mathcal{O}(V)$ with $\exp(f) = g$. This a contradiction, for example to Cauchy formula.

---

[1] Induced by the inclusion $k \to k[x_1, \dots, x_n]$

(4) First, remark that if $\mathcal{G} \in \mathrm{Sh}(U)$, then stalks of $\iota_! \mathcal{G}$ behave the following way. If $x \in U$ we have a natural isomorphism

$$(\iota_! \mathcal{G})_x \to \mathcal{G}_x,$$

and if $x \notin U$ we have $(\iota_! \mathcal{G})_x = 0$. The exactness follows from the computation at stalks. If $x \in U$ then it amounts to an isomorphism and then the zero map, and if $x \in Z$ first the zero map, and then an isomorphism.

(5) First note that

$$\mathrm{Hom}_{\mathrm{Sh_{Ab}}(X)}(j_! \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathrm{PSh_{Ab}}(X)}(j_!^{pr} \mathcal{G}, \mathcal{H}).$$

Where $j_!^{pr}$ denotes the extension by zero before sheafification. We see that a morphism $j_!^{pr} \mathcal{G} \to \mathcal{H}$ amounts to a morphism $\mathcal{G} \to j^{-1}\mathcal{H}$. Indeed if $V \not\subset U$ we have $j_!^{pr} \mathcal{G}(V) = 0$. So a map $j_!^{pr} \mathcal{G} \to \mathcal{H}$ just amounts to maps $\mathcal{G}(V) \to \mathcal{H}(V)$ which are compatible with restrictions for every $V \subset U$. In other words this exactly the data of a map of sheaves $\mathcal{G} \to j^{-1}\mathcal{H}$. This association is natural and bijective.

□

Exercise 6. *Exceptional functors (2).* We keep setup and notation as in previous exercise. Let $\mathcal{H} \in \mathrm{Sh_{Ab}}(X)$.

(1) Show that for every $s \in \mathcal{H}(V)$ for an open V, then

$$\mathrm{supp}(s) := \{x \in V \mid s_x \neq 0\}$$

is closed.

(2) Show that $\mathcal{H}_Z$, the presheaf on X defined by

$$\mathcal{H}_Z(V) = \{s \in \mathcal{H}(V) \mid \mathrm{supp}(s) \subset Z \cap V\}$$

is a sheaf. Show that $\mathcal{H}_Z(V)$ is the kernel of the map

$$\mathcal{H}(V) \to \mathcal{H}(V \cap (X \setminus Z)).$$

(3) Show that if $V' \subset V$ such that $V' \cap Z = V \cap Z$ then the restriction map $\mathcal{H}_Z(V) \to \mathcal{H}_Z(V')$ is an isomorphism.

(4) Show that for any sheaf $\mathcal{F} \in \mathrm{Sh_{Ab}}(Z)$ any map $\iota_* \mathcal{F} \to \mathcal{H}$ factors through $\mathcal{H}_Z$.

We define the *exceptional inverse image* $\iota^! \mathcal{H} := \iota^{-1} \mathcal{H}_Z$.

(5) Show that there is a natural bijection in $\mathcal{F} \in \mathrm{Sh_{Ab}}(Z)$ and $\mathcal{H} \in \mathrm{Sh_{Ab}}(X)$

$$\mathrm{Hom}_{\mathrm{Sh_{Ab}}(Z)}(\mathcal{F}, \iota^! \mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}(X)}(\iota_* \mathcal{F}, \mathcal{H}).$$

This means that for a closed immersion ι, we have a sequence of adjoints $\iota^{-1} \dashv \iota_* \dashv \iota^!$.

Solution key. (1) If $x \in X$ is such that $s_x = 0$ there is an open set around x with $s = 0$ in this open set.

(2) We show that $\mathcal{H}_Z$ is the kernel map of the unit map (so the fact that it is a sheaf will follow from this description)

$$\mathcal{H} \to j_*j^{-1}\mathcal{H}$$

which is on each open set V the restriction

$$\mathcal{H}(V) \to \mathcal{H}(V \cap U).$$

But elements s which are sent to zero by this restriction are exactly elements such that $s_x = 0$ for all $x \in V \cap U$. This happens if and only if that $\mathrm{supp}(s) \subset Z \cap V$.

(3) Let $V' \subset V$ with $V' \cap Z = V \cap Z$. It implies that

$$V = V' \cup (V \cap (X \setminus Z)).$$

We show that

$$\mathcal{H}_Z(V) \to \mathcal{H}_Z(V')$$

is an isomorphism. We show the injectivity. if s is sent to zero, then note that $s_{V'} = 0$ and $s_{V\cap(X\setminus Z)} = 0$ by construction. So $s = 0$. The surjectivity follows from gluing. If $s \in \mathcal{H}_Z(V')$ we can glue $s \in \mathcal{H}(V)$ and $0 \in \mathcal{H}(V \cap (X \setminus Z))$ to get a section of $\mathcal{H}_Z(V)$.

(4) Follows by computation at stalks at $x \in U$.

(5) Note that the presheaf preimage of $\mathcal{H}_Z$ can be expressed as, on an open set $W \subset Z$ of Z, by (the colimit ranges over opens V of X such that $V \cap Z = W$)

$$\varinjlim_{\substack{V\subset X \\ V\cap Z=W}} \mathcal{H}_Z(V).$$

Note that therefore by point (3) above this colimit is taken on isomorphisms: we mean by this that every morphism in the diagram is an isomorphism. This implies that the colimit is equal to the limit on the same system. With the fact that this colimit is taken on isomorphism we also see that this presheaf is already a sheaf.

Note first that

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}) = \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}_Z)$$

by point (4). Let W be an open of Z. A morphism $\mathcal{F} \to \iota^!\mathcal{H}$ on W amounts to a collection of morphisms $\mathcal{F}(W) \to \mathcal{H}_Z(V)$ for every $V \subset X$ open with $V \cap Z = W$ that commutes with restrictions (the colimit equals the limit). Therefore a map of sheaves $\mathcal{F} \to \iota^!\mathcal{H}$ amounts to a map for every open set $U \subset X$ of X from $\mathcal{F}(U \cap Z) \to \mathcal{H}_Z(U)$ which is compatible with every restriction. In other words, this is the data of morphism of sheaves $\iota_*\mathcal{F} \to \mathcal{H}_Z$. These identifications are natural and bijective.

□

Exercise 7. *Topological properties of schemes.* A topological space X is T_0 if for every pair of different elements $x, y \in X$ there exist an open set U of X such that exactly x or y is in U.

(1) Let X be the underlying topological space of a scheme. Show that X is T_0.

A topological space is called *irreducible* if it cannot be written as the union of two proper and non-empty closed subsets.

1. (1) Show that any non-empty open set of an irreducible topological space is dense.
2. (2) Show that if an irreducible topological space X contains at least two points, then X is not Hausdorff.
3. (3) Let A be a ring. Show that the topological space $\mathrm{Spec}(A)$ is irreducible if and only if A_{red} is an integral domain.

A topological space is called *sober* if for any non-empty irreducible closed subset $Z \subset X$, there exist a unique point $\eta_Z \in Z$ such that $\overline{\{\eta_Z\}} = Z$. In this case, we call η_Z the *generic point* of Z.

1. (1) Show that any Hausdorff topological space is sober.
2. (2) Let X be the underlying topological space of a scheme. Show that X is sober.
3. (3) Let A be an integral domain. What is the generic point of $\mathrm{Spec}(A)$?

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Solutions – week 4

Exercise 1. *Tangent vectors.* Let R be a ring, $R \to S$ an R-algebra and N an S-module. An *R-derivation*

$$d \colon S \to N$$

is an R-linear map such that for all $f, g \in S$

$$d(fg) = fd(g) + gd(f) \quad \text{(Leibniz rule)}$$

We denote this set by $\mathrm{Der}_R(S, N)$.

(1) Show that if $d \colon S \to N$ is an R-derivation, then $d(r) = 0$ for every $r \in R$.

Let $S \oplus_0 N$ denote the R-algebra with underlying R-module $S \oplus N$ and with the multiplication defined by

$$(s, n) \cdot (s', n') = (ss', sn' + s'n).$$

The multiplicative unit is $(1, 0)$.

(2) Show that the projection $S \oplus_0 N \to S$ is a ring map which has a square zero kernel ideal I, meaning that $I^2 = 0$.
(3) Show that the data of a derivation is the same as map of R-algebras $S \to S \oplus_0 N$ which is a section of the projection.

Let $X \to \mathrm{Spec}(k)$ a scheme over a field k.

(4) Show that $k[\epsilon] := k[t]/t^2$ is isomorphic to $k \oplus_0 k$.
(5) Let $x : \mathrm{Spec}(k) \to X$ be a k-rational point. We see $k(x)$ as an $\mathcal{O}_{X,x}$-algebra by the quotient map. Show that there are identifications

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x)) \cong \mathrm{Vect}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) \cong \mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$$

where $\mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$ denotes k-schemes maps[1] that sends the point of $\mathrm{Spec}(k[\epsilon])$ to x.

Solution Key. (1) We have $d(1) = d(1^2) = 2d(1)$ implying that $d(1) = 0$. Now using R-linearity $d(r) = rd(1) = 0$ for any $r \in R$.
(2) The kernel of the projection is $0 \oplus N$. Note that $(0, n) \cdot (0, n') = (0, 0)$ implying the square zero requirement.
(3) Immediate from the definition of the law.
(4) The k-algebra morphism $k[t] \to k \oplus_0 k$ sending t to $(0, 1)$ factors through $k[t]/t^2$, and is surjective. By equality of dimensions, because this map is a map of finite dimensional k-vector spaces, we deduce that this map is an isomorphism.

---

[1] meaning that the composition $\mathrm{Spec}(k[\epsilon]) \to X \to \mathrm{Spec}(k)$ is the one associated to $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ being the inclusion.

(5) First we note that as x is a k-rational point we have a k-algebra section of the surjection $\mathcal{O}_{X,x} \to k(x)$. Using this, we may write
$$\mathcal{O}_{X,x} = k(x) \oplus \mathfrak{m}_x$$
as a direct sum of k-vector spaces. Note that a k-derivation $d\colon \mathcal{O}_{X,x} \to k(x)$ will have to send the first component to zero by the first point of the exercise. Note also that if $f, g \in \mathfrak{m}_x$ then $d(fg) = f(x)d(g) + g(x)d(f) = 0$, because we have $f(x) = g(x) = 0$. Therefore, any derivation necessarily factors through $\mathfrak{m}_x/\mathfrak{m}_x^2$. It is therefore also sufficient for a k-derivation $\mathcal{O}_{X,x} \to k(x)$ to define a k-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$. This shows the first isomorphism.

For the last one, note that a k-scheme morphism from $\mathrm{Spec}(k[\epsilon])$ to X sending the point to x is equivalent to the data of a local k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$. Projecting to $k\epsilon$ and then using the identification $k \cong k(x)$, it defines a derivation $\mathcal{O}_{X,x} \to k(x)$. The other way around, given $d\colon \mathcal{O}_{X,x} \to k(x)$, we define a k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$ by (ev_x, d).

□

Remark. Note that in differential geometry, what was exposed above is a way to define tangent spaces. See *Manifolds, sheaves and cohomology by Wedhorn, Remark 5.7* Therefore, in the context of the above exercise we define
$$(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$$
to be the be the *k-tangent space* of X at x.

Remark. We have just seen that maps of k-schemes from $\mathrm{Spec}(k[\epsilon])$ into a scheme are to be interpreted as a choice of a point and one tangent vector. Therefore, we interpret $\mathrm{Spec}(k[\epsilon])$ as a point with a one dimensional tangent or *a point with an infinitesimal neighbourhood of order 1.*

Exercise 2. *Galois actions.* Let $\mathrm{Spec}(A) = X \to \mathrm{Spec}(k)$ an affine k-scheme where k is a field. Let $k \to l$ a Galois extension. We define on $\mathrm{Spec}(A \otimes_k l)$ an action of $\mathrm{Gal}(l\colon k)$ defined by $\phi_g = \mathrm{Spec}(\mathrm{id} \otimes g)$ where g is in $\mathrm{Gal}(l\colon k)$.

(1) For which $g \in \mathrm{Gal}(l\colon k)$ is the map ϕ_g a morphism of l-schemes ?
(2) Show that that the invariants[2] of $A \otimes_k l$ with this action is A.
(3) By the identification $\mathbb{C}[t] \cong \mathbb{R}[t] \otimes_{\mathbb{R}} \mathbb{C}$, show that the action of $\mathrm{Gal}(\mathbb{C}\colon \mathbb{R})$ on $\mathbb{C}[t]$ acts on the coefficients of a polynomial. For every $x \in \mathrm{Spec}(\mathbb{C}[t])$, what is $\phi_g(x)$ for the non-trivial $g \in \mathrm{Gal}(\mathbb{C}\colon \mathbb{R})$?
(4) Show that two points are identified by the natural map $\mathrm{Spec}(\mathbb{C}[t]) \to \mathrm{Spec}(\mathbb{R}[t])$ if and only if they are in the same orbit of the above action.
(5) What are the possible residue fields of points of $\mathrm{Spec}(\mathbb{R}[t])$? Show that the degree of the residue field at a closed point x as an extension of $\mathbb{R}$ corresponds to the cardinality of the fiber at x of the above map.

---

[2] Elements of $A \otimes_k l$ fixed by the action of $\mathrm{Gal}(l\colon k)$.

Solution Key. (2) Let $k \to l$ be a Galois extension. Let V be a k-vector space. We consider the base change $V_l = V \otimes_k l$. We consider the Galois action of $G = \mathrm{Gal}(l\colon k)$ on V_l given by $g \cdot (v \otimes \lambda) = v \otimes g(\lambda)$.

The goal is to show that $(V_l)^G = V$, more precisely the image of $V \to V \otimes_k l$ by $v \mapsto v \otimes 1$. We may sometimes write V for this image in what follows.

Suppose first that V is finite dimensional. Write $V = \bigoplus_i V_i$ with V_i being one dimensional. Then

$$V_l = \oplus V_{i,l} = \prod V_{i,l}.$$

and the action G restricts to this direct sum/product.

If V is one dimensional, then $(V \otimes_k l) = V$ follows from Galois theory. Note that we can compute the fixed points of a direct product as the fixed points of each components. Therefore, the case where V is finite dimensional follows: indeed $V \subset V_l^G$ with the same dimension.

To treat the arbitrary case, we want to show $(V \otimes_k l)^G \subset V$. Let $v \in (V \otimes_k l)^G$. Then there exists a finite dimensional subspace $W \subset V$ such that v is in W_l. Therefore we see by the previous case that $v \in W \subset V$, which conclude the argument.

□

Remark. By duality, it means that $\mathrm{Spec}(A)$ is the quotient of $\mathrm{Spec}(A \otimes_k l)$ by the Galois action in the category of affine schemes. This can be useful to interpret what is a scheme over an arbitrary field. Namely, schemes over algebraically closed field are more *geometric* in nature and have more simpler properties – we can therefore interpret a scheme over any field as quotient by the absolute Galois group of k of a scheme over $\overline{k}$.

Exercise 3. *Noetherian topological spaces.* We say that a topological space is *Noetherian* if there is no infinite descending sequence of closed subsets in X. Show that if A is a Noetherian ring, then $\mathrm{Spec}(A)$ is Noetherian. Does the converse hold ?

Exercise 4. *Properties of maps.* Let $A \to B$ be a ring map. Denote by $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated map of schemes.

1. Show that $A \to B$ is injective if and only $\mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ is an injective map of sheaves.
2. Show that if $A \to B$ is injective then the image of f is dense.
3. Show that if $A \to B$ is surjective then f is a closed embedding on the underlying topological spaces.

Solution key. We show that if $\varphi\colon A \to B$ is injective, then the topological image of $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is dense. Take any $a \in A$ not nilpotent. Because the map is injective $\varphi(a) \in B$ is also not nilpotent. Therefore $f^{-1}(D(a)) = D(\varphi(a)) \neq \emptyset$, implying that $D(a)$ intersects with the image of $\mathrm{Spec}(B)$, showing the claim.

□

Some notation. We introduce some notation needed for exercise 5 below. Let A and B be $\mathbb{N}$-graded rings. Let $d \geq 1$. We define the graded ring

$$A^{(d)} = \bigoplus_{n \geq 0} A_{nd}$$

and $v_d \colon A^{(d)} \to A$ for the canonical inclusion as a subring. This subring is called the d-Veronese subring.
We say that a ring map $\psi \colon A \to B$ is *homogeneous of degree* d if for $n \geq 0$ A_n maps to B_{dn}[3]. For example for the usual grading on $\mathbb{Z}[x]$ show that $x \mapsto x^d$ is homogeneous of degree d. Also, v_d is homogeneous of degree d.

Exercise 5. *Functoriality of* Proj. Let A and B be $\mathbb{N}$-graded rings. Let $\psi \colon A \to B$ be an homegeneous map of degree d for some $d \geq 1$.
To the contrary of Spec, the functoriality of Proj is not evident. The reason is that $\psi^{-1}(\mathfrak{p})$ for a prime $\mathfrak{p} \in \mathrm{Proj}(B)$ may contain the irrelevant ideal A_+.

(1) Show that
$$U(\psi) = \{\mathfrak{p} \in \mathrm{Proj}(B) \mid \psi(A_+) \not\subset \mathfrak{p}\}$$
is open. Namely show that it is the union of opens $D_+(\psi(f))$ for all homogeneous $f \in A_+$.

(2) Find an example where $U(\psi)$ is a non-empty open strict subspace of $\mathrm{Proj}(B)$.

(3) Show that ψ^{-1} defines a map of schemes $r_\psi \colon U(\psi) \to \mathrm{Proj}(A)$. Do this by defining a map $D_+(\psi((f)) \to D_+(f)$ for all homogeneous $f \in A_+$ and then glue.

(4) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is surjective then $U(\psi) = \mathrm{Proj}(B)$. Show moreover that in this case r_ψ is a topological closed embedding with image $V_+(\ker(\psi))$.

(5) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is an isomorphism then r_ψ is an isomorphism.

(6) Deduce that for any $d \geq 1$ and for $v_d \colon A^{(d)} \to A$ the map r_{v_d} is an isomorphism.

Solution key. (2) For example $\iota \colon \mathbb{Z}[x, y] \to \mathbb{Z}[x, y, z]$ the inclusion. Then $U(\iota) = D_+(x) \cup D_+(y)$. In particular $(X, Y) = [0 : 0 : 1]$ is not in this open.

(3) We use that $U(\psi)$ is a gluing
$$U(\psi) = \bigcup_{a \in A_+ \text{ hom.}} \mathrm{Spec}(B_{(\psi(a))})$$
with gluing data is given by, at the dual level of ring of functions by the natural maps

$$\begin{tikzcd}
B_{(\psi(a))} \arrow[dr] & \\
B_{(\psi(a'))} \arrow[r] & B_{(\psi(aa'))}
\end{tikzcd}$$

---

[3] Note that this means that the map factors through the d-Veronese subring.

So to define a map out of $U(\psi)$, it suffices to define a map from each $\mathrm{Spec}(B_{(\psi(a))})$. To do this we glue the map induced by ψ

$$\mathrm{Spec}(B_{(\psi(a))}) \to \mathrm{Spec}(A_{(a)}).$$

This glues because the following commutes as every map is giving by the natural extension-restriction of ψ to these rings

$$\begin{tikzcd}
A_{(a)} \arrow[r] \arrow[dr] & B_{(\psi(a))} \arrow[dr] & \\
 & A_{(aa')} \arrow[r] & B_{(\psi(aa'))} \\
A_{(a')} \arrow[r] \arrow[ur] & B_{(\psi(a'))} \arrow[ur] &
\end{tikzcd}$$

(4) We first show that $U(\psi) = \mathrm{Proj}(B)$. Let $\mathfrak{p} \in D_+(b)$ any point of $\mathrm{Proj}(B)$ and some homegeneous $b \in B_+$. Note that b^{dk_0} is in the image of ψ by hypothesis, which concludes.

To show that the map is a closed immersion, it suffices to show that locally

$$\mathrm{Spec}(B_{(\psi(a))}) \to \mathrm{Spec}(A_{(a)})$$

is a closed immersion. But therefore it suffices to show that the underlying map of rings is a surjection. Because $D_+(a) = D_+(a^N)$ for any $N \geq 1$ we can suppose that $\deg(a) \geq k_0$. But then an element of $B_{(\psi(a))}$ is of the form $\frac{b}{\psi(a)^n}$ with $\deg(b) = dn\deg(a)$. But as $A_{n\deg(a)} \to B_{dn\deg(a)}$ is surjective by assumption, we win.

Now we are left to show that the image is $V_+(\ker(\psi))$. It suffices to show that it's the image when intersecting to every $D_+(a)$. But $D_+(a) \cap V_+(\ker(\psi)) = V(\ker(\psi_{(a)}))$, which concludes.

(6) Same local trick. It suffices to show that it's locally an isomorphism. Enlarging degrees is again harmless.

□

Exercise 6. *Dimension.* Let k be a field. Compute the irreducible components and their dimension of the spectrum of the following ring

$$k[x, y, z, t]/(tx, ty, tz).$$

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Solutions – week 5

Exercise 1. *Closed subschemes of* Proj. Let B be an $\mathbb{N}$-graded ring and I be a homogeneous ideal. Show that the subset

$$V_+(I) = \{\mathfrak{p} \in \operatorname{Proj}(B) \mid I \subset \mathfrak{p}\}$$

is closed in $\operatorname{Proj}(B)$ and can be endowed a scheme structure *via* $\operatorname{Proj}(B/I)$. Show that for any $b \in B_+$ homogeneous

$$(B/I)_{(b)} \cong B_{(b)}/I_{(b)}.$$

In what follows, $V_+(I)$ is taken in the schematic sense of the previous exercise.

Exercise 2. Proj *and base change.* Let R be a ring and R' be an R-algebra. Let A be an $\mathbb{N}$-graded ring such that A_0 is an R-algebra. Define $B = A \otimes_R R'$ and write $\psi \colon A \to B$.

(1) With the notations of *week 4, exercise 5* show that $U(\psi) = \operatorname{Proj}(B)$.
(2) Show that the commutative diagram

$$\begin{tikzcd}
\operatorname{Proj}(B) \arrow[r] \arrow[d] & \operatorname{Proj}(A) \arrow[d] \\
\operatorname{Spec}(R') \arrow[r] & \operatorname{Spec}(R)
\end{tikzcd}$$

is a Cartesian square.[1]
(3) Let I be an ideal of R. Show that if $R' = R/I$ then we have $\operatorname{Proj}(B) = V_+(\bigoplus_{n\geq 0} IA_n)$ inside $\operatorname{Proj}(A)$.
(4) Let R be a ring. Let $R' = R[x_0, \dots, x_n]$. Using the above, realize $\mathbb{P}^n_R \times_R \mathbb{A}^{n+1}_R$ as the Proj of a graded ring.

Exercise 3. *Blow-ups.* Let R be a ring and $I \subset R$ an ideal. We define the blow-up of $\operatorname{Spec}(R)$ at $V(I)$ to be the map ($I^0 = R$)

$$b \colon \operatorname{Bl}_I = \operatorname{Proj}(\bigoplus_{n\geq 0} I^n) \to \operatorname{Spec}(R).$$

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[1] It means that for every scheme X and pair of morphisms $f_1 \colon X \to \operatorname{Proj}(A)$ and $f_2 \colon X \to \operatorname{Spec}(R')$ that agree when further composing to $\operatorname{Spec}(R)$, then there exists a unique morphism to $f \colon X \to \operatorname{Proj}(B)$ such that f is f_1 and f_2 when postcomposing with the maps to $\operatorname{Proj}(A)$ and $\operatorname{Spec}(R')$ respectively.

The *exceptional divisor* of the blow-up is the closed subscheme of $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$

$$E = V_+(\bigoplus_{n\geq 0} I^{n+1}).$$

(1) Show that b defines an isomorphism of schemes

$$b\colon \mathrm{Bl}_I \setminus E \to \mathrm{Spec}(R) \setminus V(I).$$

(2) Let A be a ring, $R = A[x_0, \dots, x_n]$ and $I = (x_0, \dots, x_n)$. Show that $E \cong \mathbb{P}^n_A$.

Remark. Let us introduce a bit of intuition. Points (1) and (2) subsume the key philosophy of blow-ups. First of all a blow-up is a map which is an isomorphism outside of a the fiber closed subscheme $V(I)$. We will see later that in nice cases I/I^2 is to be interpreted as the *conormal bundle* of $V(I)$ in $\mathrm{Spec}(R)$ (*i.e.* tangent vectors going out of $V(I)$). Therefore E can be interpreted as the projective space of the vector space of directions outside of $V(I)$. Meaning that for each direction going outside of $V(I)$, there is a corresponding point in E. For example, in the actual computation above, the exceptional divisor E is the space of lines through the origin in $\mathbb{A}^{n+1}$, i.e. $\mathbb{P}^n$.

(3) *Standard blow-up charts.* Consider the same setting as in the last item. Show that Bl_I can be identified as the scheme

$$V_+(x_iY_j - x_jY_i)$$

inside $\mathbb{A}^{n+1}_A \times \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \dots, x_n] \otimes_A A[Y_0, \dots, Y_n])$ where the grading is taken to be the Y-grading (see exercise 2.4).

Remark. See here for a representation of the blow-up at (x_0, x_1) of $\mathbb{A}^2$ using the standard charts. The projection to the x, y-plane is a bijection outside of the pre-image of the origin which is a line.

Solution key. (1) We show that b induces an isomorphism

$$b\colon b^{-1}(U) \to U.$$

To see this, let $f \in I$ so that $D(f) \subset U$. Note that $I_f = R_f$. Therefore by the compatibility of Proj and pullbacks we have as in the example above

$$b^{-1}(U) = \mathrm{Proj}(\bigoplus_{n\geq 0} R_f) = \mathrm{Proj}(R_f[t]) = \mathrm{Spec}(R_f).$$

As U is covered by such $D(f)$'s the above map is locally an isomorphism and therefore an isomorphism.

(2) Note that I^n/I^2 is a free A-module of rank $n+1$ generated by $x_0, \dots, x_n$. More generally I^n/I^{n+1} is a free A-a module generated by degree n-monomials. This leads to a graded isomorphism

$$\bigoplus_n I^n/I^{n+1} \cong A[x_0, \dots, x_n].$$

(3) We proof something more general.

Definition 0.1 (Regular sequence)**.** Let R be a ring. A finite sequence of elements $f_1, \dots, f_n$ is said to be a *regular sequence* if f_i is a non-zero divisor in $R/(f_1, \dots, f_{i-1})$ and $R/(f_1, \dots, f_n)$ is non-zero.

We show the following.

Proposition 0.1. *Let R be a ring and $I = (f_1, \dots, f_n)$ where $f_1, \dots, f_n$ form a regular sequence. Then the kernel of the surjection sending Y_i to $f_i^{(1)}$*

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n$$

is given by the ideal $J = (f_i Y_j - f_j Y_j)$.

Proof. We show this using two steps which both really heavily on the regular sequence hypothesis.

First, we show that the kernel of

$$R^n \to I$$

sending $e_i \mapsto f_i$ is generated by the vectors $e_i f_j - e_j f_i$. We proceed by induction on n the length of the regular sequence. For $n = 0, 1$ the claim is obvious. To proceed inductively we define the chain complex K_n

$$R^{\binom{n}{2}} \to R^n \to R$$

with R placed in degree zero and differentials being respectively given by $e_{i,j} \mapsto f_j e_i - f_j e_i$ and $e_i \mapsto f_i$. Note that $H_0(K_n)$ of this complex is $R/(f_1, \dots, f_n)$. and that the claim amounts to this complex being exact in the middle, meaning that $H_1(K_n) = 0$. Note that we also have the following exact sequence of complexes

$$\begin{tikzcd}
R^{\binom{n}{2}} \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & R \arrow[d, shift left] \\
R^{\binom{n+1}{2}} \arrow[r] \arrow[d] & R^{n+1} \arrow[r, shift left] \arrow[r, shift right, red] \arrow[d, shift right] & R \arrow[u, shift left, red] \arrow[d] \\
R^n \arrow[r] & R \arrow[r] \arrow[u, shift right, red] & 0
\end{tikzcd}$$

Where the left vertical arrows are given by $e_{i,j} \mapsto e_{i,j}$ and $e_{i,j} \mapsto \delta_{n+1,j} e_i$. The middle vertical arrows are $e_i \mapsto e_i$ and $e_i \mapsto \delta_{i,n+1}$. The first right vertical arrow is the identity.

By induction $H_1(K_n) = 0$, and we want to show that $H_1(K_{n+1}) = 0$. The long exact sequence in homology gives

$$0 = H_1(K_n) \to H_1(K_{n+1}) \to R/(f_1, \dots, f_n) \xrightarrow{\delta} R/(f_1, \dots, f_n),$$

where δ is the connecting morphism. The connecting morphism is computed by following the red arrows on the diagram above. It is therefore given by $\delta = \cdot f_{n+1}$ the multiplication by f_{n+1}. As $f_1, \dots, f_{n+1}$ is a regular sequence δ is injective and therefore $H_1(K_{n+1}) = 0$.

We have now understood the degree 1 elements of the kernel of the surjection sending Y_i to $f_i^{(1)}$

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n.$$

It now suffices to show that this kernel is generated by degree 1 elements.

Just for the rest of this proof, we call a polynomial $F \in R[Y_1, \dots, Y_n]$ to be of *weight* i if i is the minimal integer such that $F \in (Y_1, \dots, Y_i)$ but $f(Y_1, \dots, Y_n) \notin (Y_1, \dots, Y_{i-1})$. A weight 0 polynomial is defined to be 0.

This proposition will be shown as a special case (but the general case will be needed in the proof by induction) of the following.

Claim. *Let $F \in R[Y_1, \dots, Y_n]$ be an homogeneous polynomial of degree m with*

$$F(f_1, \dots, f_n) \in (f_1, \dots, f_k).$$

Then there exists an homogeneous polynomial G of degree m and weight at most k such that $F - G \in (f_iY_j - f_jY_j)$. In particular if $F(f_1, \dots, f_n) = 0$, then $F \in J = (f_iY_j - f_jY_j)$, showing the proposition.

We prove the claim by induction on the degree m of the polynomial. Let F be a polynomial of degree 1. Then

$$F(f_1, \dots, f_n) = \sum_{i=1}^{k} a_i f_i.$$

Therefore the weight k and degree 1 polynomial $G = \sum_{i=1}^{k} a_i Y_i$ satisfies the claim: indeed $F - G$ is an homogeneous polynomial of degree 1 with $(F - G)(f_1, \dots, f_n) = 0$. Therefore using the first part of the proof above, we see that $F - G \in J$.

Now if F is a polynomial of degree m, we show the claim by induction on the weight l of F. If $l \leq k$, set $F = G$. Otherwise write $F = Y_l F_1 + F_2$ with F_1 homogeneous of degree $m - 1$ and F_2 of weight at most $l - 1$. Recall that by hypothesis

$$f_l F_1(f_1, \dots, f_n) + F_2(f_1, \dots, f_n) \in (f_1, \dots, f_k) \subset (f_1, \dots, f_{l-1})$$

and $F_2(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$ because F_2 is of weight at most $l - 1$ by construction. Modding out by $f_1, \dots, f_{l-1}$ and using that $f_1 \dots, f_l$ is a regular sequence we get that $F_1(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$. We apply induction on the degree to get a polynomial G_1 of weight at most $l - 1$ such that $F_1 - G_1 \in J$. Now set $G' = Y_l G_1 + F_2$. This is a polynomial of weight at most $l - 1$ because G_1 and F_2 are. Note also that $F - G' = Y_l(F_1 - G_1) \in J$. Note also that $G'(f_1, \dots, f_n) = F(f_1, \dots, f_n) \in (f_1, \dots, f_k)$. By induction on the weight there is a polynomial G of weight at most k such that $G' - G \in J$. But now $F - G = (F - G') + (G' - G) \in J$, concluding the proof. ☐

☐

Exercise 4. *Strict transforms.* Let R be a ring. Let I be an ideal and $b \colon \mathrm{Bl}_I \to \operatorname{Spec}(R)$ the blow-up at the ideal I. Let $J \subset R$ be another ideal. We define the *strict transform of* $V(J)$ to be the blow-up of $V((I + J)/J)$ in $\operatorname{Spec}(R/J)$.

(1) Show that St_J can be identified with be the closed subscheme of Bl_I

$$V_+\left(\bigoplus_n I^n \cap J\right).$$

(2) Show that b induces an isomorphism

$$b\colon \mathrm{St}_J \setminus E \to V(J) \setminus V(I).$$

(3) *Resolving a singularity.* Let k be a field. Compute the strict transform with $R = k[x_0, x_1]$, the ideal $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$. Use the standard blow-up charts. Show that this strict transform is regular.

Solution key. (1) By definition the strict transform St_J is

$$\mathrm{Proj}(\bigoplus_{n\geq 0}(I+J)^n/J)$$

As the kernel of $I^n \to (I+J)^n/J$ is $I^n \cap J$ we see that we can realize the strict transform as the closed subscheme of Bl_I given by $V_+(\bigoplus_{n\geq 0} I^n \cap J)$.

(2) This is similar to exercise 3, point 1.

(3) Let k be a field and consider $R = k[x_0, x_1]$, $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$ which is a singular plane curve, which is the called the *node.* We compute the strict transform St_J. We claim that $\mathrm{St}_J \cong \mathbb{A}_k^1$ (which is regular) and that the blow-up map may be described as $\mathbb{A}_k^1 \to C \subset \mathbb{A}_k^2$

$$\lambda \mapsto (\lambda^2 - 1, \lambda^3 - \lambda).$$

We use the standard charts, meaning that we see $\mathrm{Bl}_I \subset \mathbb{A}^2 \times \mathbb{P}^1$. Recall that this inclusion is induced by the surjection

$$k[x_0, x_1, Y_0, Y_1] \to \bigoplus_n I^n$$

sending Y_i to x_i in degree 1. We claim that the preimage of the ideal

$$V_+\left(\bigoplus_n I^n \cap J\right)$$

by the above map is given by $(H, (x_1Y_0 - x_0Y_1))$ where

$$H = (x_1^2 - (x_0^3 + x_0^2), x_1Y_1 - (x_0^2Y_0 + x_0Y_0), Y_1^2 - (x_0Y_0^2 + Y_0^2))$$

Indeed, for degree zero, one and two, these elements are sent to the generator of J (we have $I^n \cap J = J$ for $n \leq 2$).

We now argue that these generators are enough. Note that being in I^n for a polynomial means that the monomials forming it are at least of degree n. Being in J means that the polynomial is of the form $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ for an $f(x_0, x_1) \in k[x_0, x_1]$. So we see that if such an element $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ is in $I^n \cap J$, then $f(x_0, x_1) \in I^{n-2}$ counting the degrees of the monomials because $(x_1^2 - (x_0^3 + x_0^2)) \in I^2 \setminus I^3$. Therefore for $n \geq 3$ using the degree 2 generator and elements of I in degree 1, we can attain every element of $I^n \cap J$.

Therefore the strict transform is

$$\mathrm{Proj}(A[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)).$$

Denote by B the grading ring we are taking Proj of. Note that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) = \emptyset$, so that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) \subset D_+(Y_0)$ implying that

$$\mathrm{Proj}(A[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)) = \mathrm{Spec}(B_{(Y_0)}).$$

But, if we write $\frac{Y_1}{Y_0}$ by y we get

$$B_{(Y_0)} = A[x_0, x_1, y]/(x_1^2 - (x_0^3 + x_0^2), x_1y - (x_0^2 + x_0), y^2 - (x_0 + 1), (x_0y - x_1))$$
$$\cong A[x_0, y]/(y^2 - (x_0 + 1)) \cong A[y].$$

Indeed using $x_0y = x_1$ the equation $x_1^2 - (x_0^3 + x_0^2)$ turn into $x_0^2(y^2 - (x_0 + 1))$ and $x_1y - (x_0^2 + x_0)$ turn into $x_0(y^2 - (x_0 + 1))$ which are both subsumed by the equation coming from degree 2.

Therefore we see that the strict transform is isomoprhic to $\mathbb{A}^1$. By using that under this isomorphisms $x_0 \mapsto y^2 - 1$ and $x_1 \mapsto x_0y = y^3 - y$ the claim about the form of the map follows.

□

Remarks. The equation of the last item is the equation of a *nodal curve* which is a type of singular curve. See here for a representation. The tangent space at the origin has dimension 2, which is why the curve is not regular. Since the blow-up at a point replaces a point by “directions out of the point”, it is no surprise that blowing up a node at its nodal point removes the singularity.

Exercise 5. *A criterion for affineness.* Let X be a scheme. Let $f \in \mathcal{O}_X(X)$. Let

$$X_f = \{x \in X \mid f(x) \neq 0\}$$

where $f(x)$ denotes the image of f in $k(x)$.

1. Show that X_f is open.
2. Show the following lemma.
   Lemma. *A scheme X is affine if and only if there exists $f_1, \ldots, f_n \in \mathcal{O}_X(X) = A$ that generates the unit ideal in A and opens X_{f_i} are affine.*

For item (2), show that it suffices to show that the natural map

$$\Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$$

is an isomorphism.
To show this, show the following intermediate lemmas.

(a) For injectivity, show that if X is a quasi compact scheme and $g \in \Gamma(X, \mathcal{O}_X)$ such that the restriction of g to $\Gamma(X_f, \mathcal{O}_{X_f})$ is zero, then there is some $n > 0$ such that $f^n g = 0$ in $\Gamma(X, \mathcal{O}_X)$.

(b) For surjectivity, show that if X admits a finite cover by affine schemes with intersections being quasi-compact, then if $g \in \Gamma(X_f, \mathcal{O}_{X_f})$ then there is some $n > 0$ such that $f^n g$ is the image of an element $\Gamma(X, \mathcal{O}_X)$.

Solution key. We show the two lemmas mentioned in the exercise.

(a) We first treat the affine case $\mathrm{Spec}(A)$. If $g \in A$ is zero in A_f the by definition of localization we get that there is some n such that $f^n g = 0$. Now if X is quasi-compact, we cover X by finitely many open affines $(U_i)_{i=1}^n$ and find n_i such that $f^{n_i} g$ is zero when restricited to the respective open affines. Taking $N = \max\{n_i\}$ concludes by the sheaf property.

(1) We also first treat the affine case $X = \mathrm{Spec}(A)$. If $g \in A_f$ the there is some $n > 0$ such that $f^n g$ is the image of an element $a \in A$. Now if X admits a cover as in the hypothesis, then we can find a_i in $\Gamma(U_i, \mathcal{O}_{U_i})$ such that a_i restricted to $U_{i,f}$ is $f^{n_i} g$. Arranging n and a_i by suitably multiplying by power of f to get that such that a_i restricts to $f^n g$ on $U_{i,f}$. Now, $a_i - a_j \in \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$ restricts to zero on $(U_i \cap U_j)_f$. Therefore by item (a), there is some n_{ij} such that $f^{n_{ij}}(a_i - a_j) = 0$ in $\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$. Again, up to multiplying by some suitable power N we can replace a_i's so that they are compatible on intersections. Therefore by the sheaf property there is some $a \in \Gamma(X, \mathcal{O}_X)$ which satisfies what we want.

□

Exercise 6. Let k be an algebraically closed field. Are the following schemes regular?

(1) $V_+(XZ - Y^2) \subseteq \mathbb{P}^2$
(2) $V(xz - y^2) \subseteq \mathbb{A}^3$
(3) $V_+(XZ - Y^2, YW - Z^2, XW - YZ) \subseteq \mathbb{P}^3$
(4) $V(y^2 - x(x-1)(x+1) \subseteq \mathbb{A}^2$

Hint: Be careful about the characteristic of k!

Exercise 7. Let $k = \mathbb{F}_p(t)$, and consider $X_k = V(x^p - t) \subseteq \mathbb{A}^1_k$. Show that X is regular.
However, let $k' = \mathbb{F}_p(t^{1/p}) \supseteq k$. Then show that $X_{k'}$ is not regular.

Remark. As we will shortly see, we have a Cartesian diagram

$$\begin{tikzcd} X_{k'} \arrow[r] \arrow[d] & X_k \arrow[d] \\ \mathrm{Spec}(k') \arrow[r] & \mathrm{Spec}(k). \end{tikzcd}$$

We say that $X_{k'}$ is obtained by a *base field extension* (or *base change*) of X_k.
What the previous exercise then tells us is that that being regular (or even reduced) is not stable under field extensions. We will later see a notion, called *smoothness*, which implies regularity, and which is stable under base change.

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AG II - Schemes and sheaves

Solutions – week 6

Exercise 1. *Normal schemes and normalization* An integral scheme X is said to be *normal* if every stalk $\mathcal{O}_{X,x}$ is integrally closed.

(1) Show that an affine integral scheme $\mathrm{Spec}(A)$ is normal if and only if A is normal ring.

(2) Show that an integral scheme is normal if and only for every closed point $x \in U$ the stalk $\mathcal{O}_{X,x}$ is integrally closed for every open affine $U \subset X$.[1]

The *normalization* of an integral scheme X is a scheme $\widetilde{X}$ together with a dominant map[2] $\nu\colon \widetilde{X} \to X$ such that for every dominant morphism from an integral normal scheme $f\colon Z \to X$ there exists a unique morphism $\overline{f}\colon Z \to \widetilde{X}$ with $\nu\overline{f} = f$. Therefore the normalization is unique up to unique isomorphism.

(3) Let A be an integral domain. Show that if $X = \mathrm{Spec}(A)$, then $\mathrm{Spec}(\widetilde{A})$ is the normalization of X if $A \to \widetilde{A}$ is the normalization of A.

(4) Show that every integral scheme admits a normalization.

Solution key. We first remark the following general fact about integral domains

$$A = \bigcap_{\mathfrak{m}\in\max(A)} A_{\mathfrak{m}}.$$

Indeed, if $x \in \bigcap_{\mathfrak{m}\in\max(A)} A_{\mathfrak{m}}$ the ideal

$$I_x = \{a \in A \mid ax \in A\}$$

needs to contain 1, implying that $x \in A$. Otherwise there is some maximal ideal $\mathfrak{m} \supset I_x$. But as we can write $x = a\lambda^{-1}$ with $a \in A$ and $\lambda \in A \setminus \mathfrak{m}$, we get that $\lambda \in I_x$ a contradiction.

(1) and (2)
Now suppose that for every maximal ideal $\mathfrak{m}$ the local ring $A_{\mathfrak{m}}$ is normal. Write $K = \mathrm{Frac}(A)$. If $a \in K$ is the root of a monic polynomial in $A[t]$, it is therefore also the root of the same monic polynomial seen in $A_{\mathfrak{m}}[t]$, implying that $a \in A_{\mathfrak{m}}$. The above implies that $a \in A$ and as a byproduct, A is normal.
For the converse, we show that any localization of an integral normal ring is again normal. Say S is a multiplicative subset. Take $x \in K$ to be a root of a monic polynomial in $S^{-1}A[t]$. Clearing the denominators and multiplying

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[1] For finite type k-schemes, this the same as saying every closed point of X. See week 10, exercise 1.

[2] A map is called *dominant* if the topological image of the map is dense.

by enough elements of s, we see that there is an $s \in S$ such that sx is a root of a monic polynomnial in $A[t]$, implying that $sx \in A$, and that $x \in S^{-1}A$.
(3) Note that first that if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is dominant with A reduced, it implies that $A \to B$ is injective. Indeed, if $a \mapsto 0$, it implies that $D(a)$ does not meet the image. But then, $D(a) = \emptyset$, implying that a is nilpotent. As A is reduced, the claim follows.
Now the universal property in the category of affine schemes amounts to check equals by duality to the following. If B is normal, and $A \to B$ is injective, then there is a unique factorization $A \to \widetilde{A} \to B$. Consider $K_A \to K_B$ the induced map. If $x \in K_A$ is the root of a monic polynomial in $A[t]$, the image in K_B is the root of the same polynomial seen in $B[t]$, implying that the image is in B. This concludes.
Now we prove that the universal property also holds in the category of schemes. Let $f\colon Z \to \mathrm{Spec}(A)$ be any dominant map from an integral normal scheme. Cover Z by affine, necessearly normal integral, schemes (Z_i). Then $f_i\colon Z_i \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(\widetilde{A})$ by the above. By the universal property, it glues to a necessarily unique map $\widetilde{f}\colon Z \to \mathrm{Spec}(\widetilde{A})$.
(4)

- First we make the important remark for the construction that normalization preserves open immersions. More precisely, if $A \to A'$ is an affine map between integral domains that induces an open immersion, then $\widetilde{A} \to \widetilde{A'}$ also induces an open immersion. The key is that if S is a multiplicative subset of A, we have $\widetilde{S^{-1}A} = S^{-1}\widetilde{A}$. Using this we show the claim. That $A \to A'$ is an open immersion means that there exists a finite number of functions $a_1, \dots a_n \in A$ such that the localization $A_{a_i} \to A'_{a_i}$ is an isomorphism, and the image of the a_i's generated the unit ideal in A'. Using that we can commute localization and normalization as stated above, we get that the maps $\widetilde{A}_{a_i} \to \widetilde{A}'_{a_i}$ are also isomorphisms also, showing the claim.
- Now we show that any separated integral scheme admits a normalization. Say X is such a scheme, and that X is covered by affine schemes X_i's with affine intersection (by separated) X_{ij}. We claim that we can glue the schemes $\widetilde{X}_i$'s to a scheme $\widetilde{X}$ together with a map $\widetilde{X} \to X$. By the above the image of $\varphi_{ij,i}\colon \widetilde{X}_{ij} \to \widetilde{X}_i$ is open. We write it $U_{i,j}$. Now note that $\varphi_{ij,j}\varphi_{ij,i}^{-1}\colon U_{i,j} \to U_{j,i}$ is an isomorphism. We denote this last map $\psi_{i,j}$. Using the universal property of $\widetilde{-}$ in the affine case, it follows that $(\psi_{i,j})_{(i,j)}$ is a collection that satisfies the cocyle condition, allowing us to proceed to the usual gluing construction. Note that the maps $\widetilde{X}_i \to X_i$ glue by construction to a map $\widetilde{X} \to X$. To show that this has the required universal property, let $f\colon Z \to X$ be any dominant map from an integral normal scheme. Write $Z_i = f^{-1}(X_i)$. By the above affine case there is a unique map $\widetilde{f}\colon Z_i \to \widetilde{X}_i \to \widetilde{X}$ which glues to a necessarily unique map $f\colon Z \to \widetilde{X}$ showing the claim.
- The general case follows by the same pattern and the separated case. Namely, any scheme can be covered by an union of open separated (affine) schemes such that the intersection is separated.

□

Exercise 2. *Blow-ups.* Let k be an algebraically closed field. You can use the following.
Let $A = k[x_1, \dots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of f with respect to x_i. Then

$$\operatorname{Spec}(A) \text{ is regular } \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

Moreover $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of $\operatorname{Spec}(A)$*.*

(0) Let R be a ring. Show that if $I = (f_0, \dots, f_n)$ is generated by a regular sequence then $\mathrm{Bl}_I = V_+(X_i f_j - X_j f_i)$ in $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec}(R)$. (Use the lemmas in the blow-ups document from moodle)
(1) Show that blow-up of (x^2, y^2) in $\operatorname{Spec}(k[x,y])$ is not normal and that the blow-up of (x,y) is its normalization.
(2) Show that blow-up of (x^2, y) in $\operatorname{Spec}(k[x,y])$ is not regular. What are the regular points?[3]
(3) Show that $X = \operatorname{Spec}(k[x,y,z,w]/(xy - zw))$ is not regular. What are the regular points?
(4) Show that blow-ups of X at (x,y,z,w) and (x,z) are regular. We denote these blow-ups by $X_1 \to X$ and $X_2 \to X$.
Remark. This is another example where blow-ups resolves (=removes) singularities, as in 4.(3) of week 5.
(5) Compute fibers of (x,y,z,w) of $X_1 \to X$ and $X_2 \to X$.

Solution key. This exercise was a previous year hand-in exercise so solutions are credited to past students of the course who wrote them.

(1)(Joel) Let $A = k[x,y]$, $I = (x^2, y^2)$ and $R = A/I$. Consider the map $\phi : A[Z,W] \twoheadrightarrow \bigoplus_{n \geq 0} I^n$ which sends $Z \to x^2$ and $W \to y^2$ in degree one. Then $\ker \phi = (Zy - Wx)$, so $\mathrm{Bl}_I \cong \operatorname{Proj}\ A[Z,W]/(Zy^2 - Wx^2)$. Next, we show that the blow-up is not normal. Consider the affine chart U_W where $W \neq 0$, which is given by $\operatorname{Spec} k[x,y,z]/(zy^2 - x^2) =: \operatorname{Spec} B$, where $z = \frac{Z}{W}$. Then $\frac{zy}{x} \in \operatorname{Frac}(B)$, and $\left(\frac{zy}{x}\right)^2 = \frac{z \cdot zy^2}{x^2} = \frac{zx^2}{x^2} = z$. Hence, $\frac{zy}{x}$ is a root of the monic polynomial $P(t) = t - z$ with coefficients in A. Now $\frac{zy}{x} \notin B_{((x,y))}$, as x is not inverted in the localization, and the field of fractions of $B_{((x,y))}$ is the same as for B, we see that the blow-up is not normal.

The affine chart U_W can also be expressed as $\operatorname{Spec} k[x, y, \frac{x^2}{y^2}]$, and similarly we get a chart $U_Z \cong \operatorname{Spec} k[x, y, \frac{y^2}{x^2}]$. As above, neither of these affine charts are normal, and we can normalize on the ring level by exercise (2). Hence, for $k[x, y, \frac{x^2}{y^2}]$ the normalization is given by $k[x, y, \frac{x^2}{y^2}][\frac{x}{y}] = k[x, y, \frac{x}{y}] = k[y, \frac{x}{y}] = k[y, t]$, and similarly for we get $k[x, t']$ as the normalization for the ring corresponding to U_Z. Thus we have two affine planes over k as the normalizations of our charts.

---

[3] This investigation can be used to show that this blow-up is normal.

Let us inspect the blow-up of $J = (x, y)$: the blow-up algebra Bl_J is isomorphic to $\tilde{A} := k[x, y][Z, W]/(xW - yZ)$ by the same procedure as in the beginning. Here, we have the charts $\tilde{U}_W = \text{Spec } k[x, y, z']/(x - yz') \cong \text{Spec } k[x, y, \frac{x}{y}] = \text{Spec } k[y, \frac{x}{y}] = k[y, t]$ when $W \neq 0$ and similarly $\tilde{U}_Z = \text{Spec } k[x, y, w']/(x - yw') \cong \text{Spec } k[x, t']$, which are the normalizations of the two affine charts of the blow up of (x^2, y^2). Now, on the intersection $U_Z \cap U_W$ of Proj Bl_I we have $Z, W \neq 0$, so $U_Z \cap U_W \cong k[x, y, \frac{x^2}{y^2}, \frac{y^2}{x^2}]$, with its normalization given by $k[x, y, \frac{x}{y}, \frac{y}{x}] = k[y, \frac{x}{y}, \frac{y}{x}]$, which corresponds to the intersection on the blow up of (x, y). Hence, we can glue to get the normalization of the blow-up of (x^2, y^2).

(2)(Joel) Let $I = (x^2, y)$ and $R = A/I$. The blow-up is isomorphic to $\mathrm{Bl}_I \cong \text{Proj } A[Z, W] = \text{Proj } \; k[x, y][Z, W]/(yZ - x^2W)$, which we can cover with $U_Z = \text{Spec } k[x, y][a]/(y - ax^2)$ and $U_W = \text{Spec } k[x, y][b]/(by - x^2)$, where $a = \frac{W}{Z}$ and $b = \frac{Z}{W}$. On U_W we see that at the point $x = b = 0$ the scheme is not regular by the criterion provided in the exercise, as $(0, 0, 0) \in V(by - x^2, -2x, b, y)$. This is the only non-regular point, as on U_Z we have $V(y - ax^2, -2a, 1, -x^2) = \varnothing$. Hence all points except $x = y = W = 0$ are regular.

(3)(Julie) Let $g = xy - zw \in k[x, y, z, w]$. By the criterion provided in the statement of the exercise, the set of non-regular points in

$$\text{Spec } \left(\frac{k[x, y, z, w]}{(xy - zw)}\right)$$

is given by

$$V(g, \partial_x g, \partial_y g, \partial_z g, \partial_w g) = V(xy - zw, y, x, -w, -z) = V(x, y, z, w) = \{(x, y, z, w)\},$$

where the last equality holds by maximality of (x, y, z, w) in $k[x, y, z, w]$. Hence, all points of X are regular except for (x, y, z, w) (corresponding to the origin in $V(xy - zw) \subseteq \mathbb{A}^4_k$).

(4)(Maxence) Consider $R = k[x, y, z, w]$. Let $I = (x, y, z, w)$, $I' = (x, z)$ and $J = (xy - zw)$. We consider the strict transform St_J (resp. St'_J) of $V(J) = X$ at I (resp. I') in $\mathbb{A}^4_k$. We denote these schemes as respectively X_1 and X_2. We know that X_1 (resp. X_2) is the closed subscheme $V_+(\bigoplus_n I^n \cap J)$ of Bl_I (resp. the closed subscheme $V_+(\bigoplus_n I'^n \cap J)$ of $\mathrm{Bl}_{I'}$).

Notice that $\mathrm{Bl}_I = \mathrm{Proj}(R[X, Y, Z, W]/\tilde{I})$ and $\mathrm{Bl}_{I'} = \mathrm{Proj}(R[X, Z]/\tilde{I}')$ where

$$\tilde{I} = (yX - xY, zX - xZ, wX - xW, yZ - zY, yW - wY, zW - wZ) \text{ and } \tilde{I}' = (zX - xZ).$$

So, the preimage of the ideal $\bigoplus_n I^n \cap J$ by the natural surjection is given by the ideal $K = \tilde{I} + (xy - zw, xY - zW, XY - ZW)$. Indeed, it must be generated in $R[X, Y, Z, W]$ by homogeneous polynomials with degree less or equal to 2 with respect to the variables X, Y, Z, W whose image is send to the generator of J which has degree 2. These generators are enough since every elements f in I^n has monomials of at least degree n and if $f \in J$, then $f = g \cdot (xy - zw)$. Since $xy - zw$ is of degree 2, the polynomial g must be of degree $n-2$, hence $g \in I^{n-2}$. So for every element in $I^n \cap J$ with $n \geq 3$ can be reach using generators of K. In the same way the preimage of $\bigoplus_n I'^n \cap J$ by the natural surjection is the ideal $K' = \tilde{I}' + (xy - zw, yX - wZ)$.

That is,

$$X_1 = \operatorname{Proj}(R[X,Y,Z,W]/K) \text{ and } X_2 = \operatorname{Proj}(R[X,Z]/K').$$

For X_1 on $D_+(X)$, we have $\mathcal{O}_{X_1}(D_+(X)) = k[x, s_1, s_2, s_3]/(s_1 - s_2 s_3)$ by simplifying the equations of K. And by the criterion, the affine open subset $D_+(X)$ of X_1 is regular. The same result holds for $D_+(Y)$,$D_+(Z)$ and $D_+(W)$ by symetry of the variables. Hence X_1 is regular.

For X_2 on $D_+(X)$, we have $\mathcal{O}_{X_2}(D_+(X)) = k[x, w, s]$ by simplying equations of K', and so $D_+(X) = \mathbb{A}^3_k$ which is regular. The same result holds for $D_+(Z)$ by symetry. Hence X_2 is regular.

(5)(Maxence) We want to compute the fiber of $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ over (x,y,z,w).

First, the residue field of $(x,y,z,w) \in X$ is simply k by exactness of localization, so for $i = 1, 2$, we need to compute the fibred product $X_i \times_X \operatorname{Spec}(k)$. Hence, if we denote $A = k[x,y,z,w](xy - zw)$ we have

$$X_1 \times_X \operatorname{Spec}(k) = \operatorname{Proj}\left(\frac{A[X,Y,Z,W]}{K} \otimes_A k\right) \text{ and } X_2 \times_X \operatorname{Spec}(k) = \operatorname{Proj}\left(\frac{A[X,Z]'}{K} \otimes_A k\right).$$

Looking at these tensor products, by using A-linearity all relations given by K vanish except $XY - ZW = 0$ in the residue field of (x,y,z,w) by its definition. The same holds for K' but here all its relations vanish.

It yields that

$$X_1 \times_X \operatorname{Spec}(k) = \operatorname{Proj}(k[X,Y,Z,W]/(XY - ZW)) = \mathbb{P}^1_k \times_{\operatorname{Spec}(k)} \mathbb{P}^1_k$$

and

$$X_2 \times_X \operatorname{Spec}(k) = \operatorname{Proj}(k[X,Z]) = \mathbb{P}^1_k.$$

□

Exercise 3. *Integrality/reducedness of* Proj. Let B be an $\mathbb{N}$-graded integral/reduced ring. Show that $\operatorname{Proj}(B)$ is an integral/reduced scheme.

Solution key. If B is reduced any localization is also. Therefore the degree zero part of any localization by homogeneous elements are also. It implies that $\operatorname{Proj}(B)$ is reduced. If B is integral, the product ss' of two non-zero homegeneous elements s, s' is never zero. It implies that the degree zero part of $B_{ss'}$ is not zero also. It implies that the intersection of two non-empty opens is never empty in $\mathrm{Proj}(B)$. Therefore $\mathrm{Proj}(B)$ is irreducible. Being also reduced, it is integral. □

Exercise 4. *Fibers.*

(1) Compute the fibers of the morphism
$$\mathrm{Spec}(\mathbb{Z}[x,y,z]/(2zx+9y^2)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?

(2) Compute the fibers of the morphism, where p is a prime number
$$\mathrm{Spec}(\mathbb{Z}[x,y]/(xy^2+p)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?

Solution key. (1) The fiber over 2 is not reduced. The fiber over 3 is reduced but not integral. It is integral over any other prime by Eisenstein criterion.

(2) The fiber over p is not reduced and not irreducible. Othewise it is isomorphic to $k[x,y,y^{-1}]$ where k is a prime field not equal to $\mathbb{F}_p$. □

Exercise 5. *Properties under base change.* Let $f\colon X \to Y$ be a morphism of schemes. Which of the following properties are stable under base change? Prove the statement or provide a counter-example.

(1) f is an open immersion.
(2) f is a closed immersion.
(3) f is injective.
(4) f has integral fibers.
(5) f has reduced fibers.

Solution key. Statements (1) and (2) are true (proof below), for (3) take $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ while a counter example to the remaining is the map $\mathrm{Spec}(\mathbb{F}_p(t^{1/p})) \to \mathrm{Spec}(\mathbb{F}_p(t))$, base changed against itself.

Let us start with open immersions. Up to composing by an isomorphism we can suppose that $f\colon X \to Y$ is $U \subset Y$ an open.

But now we see that the following is a pullback diagram
$$\begin{tikzcd} U' \arrow[r] \arrow[d, "f'"'] & U \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Let $U' = g^{-1}(U) \subset Y'$ open, equipped with the open-subscheme of Y'-structure. Indeed the universal property of the pullback here reads as a map $Z \to Y'$ that topologically factors to the open $f^{-1}(U)$, implying that it factors schematically because the sheaf on the open is just the restriction of the sheaf on the all set.

We now prove and (2). First, a map $f\colon X \to Y$ is a closed immersion if and only if $f\colon f^{-1}(U_i) \to U_i$ is a closed immersion for $\bigcup U_i = Y$ an open cover. Indeed a subset $Z \subset Y$ is closed if and only if $U_i \cap Z \subset U_i$ is closed for every

i and to check that the desired sheaf map is surjective is and only if it is locally.
Therefore if

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

is a pullback diagram with f being a closed immersion, we can reduce to the affine case as follows. First, take a cover (U_i) of Y by affines, and consider the cover of X induced by the pre-images $(g^{-1}(U_i))$. Then cover each of these opens $g^{-1}(U_i)$ by affines (V_{ij}). Then

$$\begin{tikzcd}
f'^{-1}(V_{ij}) \arrow[r] \arrow[d, "f'"] & f^{-1}(U_i) \arrow[d, "f"] \\
V_{ij} \arrow[r] & U_i
\end{tikzcd}$$

is again a pullback diagram.
We now use the following lemma.
Lemma. *Let* $X = \mathrm{Spec}(A)$ *be affine and* $\iota\colon Z \to X$ *a closed immersion. Then the natural map* $Z \to \mathrm{Spec}(\mathcal{O}_Z(Z))$ *is an isomorphism and*

$$A \to \mathcal{O}_Z(Z)$$

is surjective. If I *is the kernel of this map, we therefore have*

$$\begin{tikzcd}
Z \arrow[r, "\iota"] \arrow[d, "\sim"'] & \mathrm{Spec}(A) \\
\mathrm{Spec}(A/I) \arrow[ur] &
\end{tikzcd}$$

Proof. Let $Z = \cup_i V_i$ a finite covering by affines. By hypothesis $V_i = U_i \cap Z$ for some open U_i of X. Covering all U_i and $X \setminus Z$ by finitely many principal opens of X we can suppose that $V_i = D(f_i) \cap Z$ for some $f_i \in \mathcal{O}_X(X)$ with (f_i) being the unit ideal in A, and therefore in $\mathcal{O}_Z(Z)$ also. Now we use week 5.5.2 to conclude that Z is affine. Therefore $Z \to \mathrm{Spec}(\mathcal{O}_Z(Z))$ is an isomorphism.
By assumption for every $\mathfrak{p} \in \mathrm{Spec}(A)$ the map $\mathcal{O}_{X,\mathfrak{p}} \to (\iota_*\mathcal{O}_Z)_{\mathfrak{p}}$ is surjective. When $\mathfrak{p} \notin Z$ the right is zero and coincides with $\mathcal{O}_Z(Z)_{\mathfrak{p}}$: indeed take $\mathfrak{p} \in D(f_i) \subset X \setminus Z$, then as $D(f_i) \cap Z = \emptyset$, we conclude that f_i in $\mathcal{O}_Z(Z)$ is nilpotent and as $\mathcal{O}_Z(Z)_{\mathfrak{p}}$ is a further localization of $\mathcal{O}_Z(Z)_{f_i} = 0$ we have our claim. When $\mathfrak{p} \in Z$ the right hand side is $\mathcal{O}_{Z,\mathfrak{p}}$, and because X and Z are affine this is $A_{\mathfrak{p}} \to \mathcal{O}_Z(Z)_{\mathfrak{p}}$. So we conclude that $A \to \mathcal{O}_Z(Z)$ is a map of A-modules surjective at every localization at primes, implying that this map is surjective. □

Therefore $f^{-1}(U_i)$ is also affine. Because the inclusion of affine schemes into schemes preserve limits, we are therefore in a situation

$$\begin{tikzcd}
\mathrm{Spec}(B \otimes_A A/I = B/IB) \arrow[r] \arrow[d, "f'"] & \mathrm{Spec}(A/I) \arrow[d, "f"] \\
\mathrm{Spec}(B) \arrow[r, "g"] & \mathrm{Spec}(A)
\end{tikzcd}$$

which concludes.

□

Exercise 6. *An open of an affine is not neccesarly affine.* Let R be a non-zero ring. Show that $U = \mathrm{Spec}(R[x, y]) \setminus V(x, y)$ is not affine.
Hint: compute $\mathcal{O}(U)$ using an appropriate cover and the sheaf property.

Solution key. We use the cover $D(x) \cup D(y)$ and the sheaf property to compute global sections of U. Because x, y are non zero divisors, localization maps $R[x^{\pm 1}, y] \to R[x^{\pm 1}, y^{\pm 1}]$ and $R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$ ar injective and we may treat them as inclusions. Now, global sections are the elements of the kernel of the map

$$R[x^{\pm 1}, y] \times R[x, y^{\pm 1}] \to R[x^{\pm 1}, y^{\pm 1}]$$

that sends $(f, g) \mapsto f - g$. In other words

$$\mathcal{O}(U) = R[x^{\pm 1}, y] \cap R[x, y^{\pm 1}] = R[x, y].$$

If U was affine, then the natural $U \to \mathrm{Spec}(R[x, y])$ would be an isomorphism, because it an inclusion of an open, an equality. But because $R \neq 0$, $\mathrm{Spec}(R[x, y] \setminus U) = \mathrm{Spec}(R)$ is non empty, a contradiction.

□

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AG II - Schemes and sheaves

Solutions – week 7

Exercise 1. *Finite covers of* $\mathbb{P}^1_{\mathbb{C}}$. Let $n \geq 1$. Consider the self map c_n of $\mathbb{C}$-schemes on $\operatorname{Proj}(\mathbb{C}[x,y]) = \mathbb{P}^1_{\mathbb{C}}$ induced by Proj from the $\mathbb{C}$-algebra map $x \mapsto x^n$ and $y \mapsto y^n$ on $\mathbb{C}[x,y]$.

(1) Compute the preimage by this map of $D_+(x)$ and $D_+(y)$, show it's affine and show that the induced map of rings at global sections $c_n^{-1}(D_+(x)) \to D_+(x)$ (and same with y) is finite.[1]

(2) Compute all the fibers of the map.

Solution key. (1) The preimage of $D_+(x)$ and $D_+(y)$ is $D_+(x^n) = D_+(x)$ and $D_+(y^n) = D_+(y)$. The map on ring of functions is given the $\mathbb{C}$-algebra map given by sending $\frac{y}{x} \mapsto \left(\frac{y}{x}\right)^n$ and $\frac{x}{y} \mapsto \left(\frac{x}{y}\right)^n$. Now note that $\mathbb{C}[t^n] \subset \mathbb{C}[t]$ is finite free of degree n. A basis being $1, t, \dots, t^{n-1}$.

(2) The generic fiber is $\mathbb{C}(\frac{x}{y})$ because the generic point's unique preimage is the generic point for dimension reasons. But let us also deduce as follows. Locally this amounts to compute the tensor product=pushout of

$$\begin{tikzcd}
\mathbb{C}[x/y] \arrow[r, "x/y \mapsto (x/y)^n"] \arrow[d] & \mathbb{C}[x/y] \\
\mathbb{C}(x/y) &
\end{tikzcd}$$

But note that localizing $\mathbb{C}[x/y]$ at the multiplicative $(\mathbb{C}[x/y])^n \setminus 0$ (where the power n here means elements of this ring that are the n-th power) is the same as localizing by the multiplicative subset $\mathbb{C}[x,y] \setminus 0$. Indeed inverting an element or it's n-th power is the same. As for the fiber of closed points, say $(t - \lambda)$ where $t = x/y$ or y/x seen in $D_+(x)$ or $D_+(y)$, we get

$$\operatorname{Spec}(\mathbb{C}[t]/(t^n - \lambda)).$$

So if $\lambda = 0$, we get a single non-reduced point and n-copies of $\operatorname{Spec}(\mathbb{C})$ otherwise.

□

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[1] A map of rings $A \to B$ is finite if B is a finitely generated A-module. This implies that this self map is a *finite map of schemes* a notion to be introduced in the lecture soon.

Exercise 2. *Cones.* Let S be a graded ring finitely generated in degree 1. We define the *Cone* of $\mathrm{Proj}(S)^2$ to be the $\mathrm{Spec}(S)$. We call $v := V(S_+)$ the vertex of the cone.

(1) Consider $T = \mathrm{Bl}_v(\mathrm{Spec}(S))$. Suppose that S is generated in degree 1. Show that the blow-up algebra (see week 5, exercise 3) is

$$S' = \bigoplus_{n \geq 0} \left(\bigoplus_{k \geq n} S_k \right).$$

The aspect *generated in degree 1* is crucial for the above. We suppose this for the rest of the exercise.

(2) Show that the natural graded map $S \to S'$ induces a morphism $f \colon T \to \mathrm{Proj}(S)$.

(3) Show that f restricted to the exceptional divisor of the blow-up is an isomorphism.

(4) Let a be an element of degree 1 in S. Show that $f^{-1}(D_+(a)) \cong \mathrm{Spec}(S_{(a)}[t])$.

Solution key. (1) Note that as the algebra is finitely generated in *degree 1*, we have that for $n \geq 1$

$$(S_+)^n = \bigoplus_{k \geq n} S_k.$$

The result follows.

(2) Just note that S_0-generators in degree 1 are sent to S-generators in degree 1.

(3) Note that $S_+^n / S_+^{n+1} = S_n$. Therefore, the result follows.

(4) Let $a \in S_1$ be a degree 1 element. We denote by B the blow-up algebra we are working with, and if $f_i \in S_i$ then we denote by $f_{i,(k)} \in B_k$ for this element placed in degree $i \geq k$.

We first remark that $S_{(a)}$ embeds as a ring in $B_{(a_{(1)})}$ by

$$\frac{f_k}{a^k} \mapsto \frac{f_{k,(k)}}{a^k_{(1)}}.$$

Note also that any element in $B_{(a_{(1)})}$ is a sum of elements of the form

$$\frac{f_{i,(k)}}{a^k_{(1)}}$$

where $f_i \in S_i$ with $i \geq k \geq 0$. Now note that in B, if $d = i - k$, we have

$$f_{i,(k)} a^d_{(1)} = f_{i,(i)} a^d_{(0)} = (f_i a^d)_{(d+k)}.$$

In consequence we can express the above as

$$\frac{f_{i,(k)}}{a^k_{(1)}} = \left(\frac{f_{i,(k)}}{a^i_{(1)}} \right) a^d_{(1)} = \left(\frac{f_{i,(i)}}{a^i_{(1)}} \right) a^d_{(0)}.$$

---

[2] Note that the following algebra depends on the algebra S, and so not only on the isomorphism class of $\mathrm{Proj}(S)$.

In consequence the $S_{(a)}$-algebra map $S_{(a)}[t] \to B_{(a_{(1)})}$ sending to $t \mapsto a_{(0)}$ is surjective. We claim that it is also injective. Indeed, suppose that (here each i in the sum depends on d but we omit it to simplify the notation)

$$0 = \sum_{d=0}^{n} \left(\frac{f_{i,(i)}}{a_{(1)}^{i}} \right) a_{(0)}^{d}$$

Then, there is a $N \geq 0$ such that in $(B)_N = \bigoplus_{k \geq N} S_k$ we have

$$0 = \sum_{d=0}^{n} \left(f_{i,(i)} a_{(1)}^{N-i} \right) a_{(0)}^{d} = \sum_{d=0}^{n} \left(f_i a^{N-i+d} \right)_N$$

But each individual term of this sum is of degree (seen as elements of S) $d+N$. Therefore, the claim follows.

Remark. Note that we proved more precisely that $B_{(a_{(1)})}$ identifies with

$$\bigoplus_{d \geq 0} \widetilde{S}(d)(D_+(a)).$$

We see therefore that the blow-up of the cone at vertex identifies with $\mathbb{V}(\mathcal{O}_{\mathrm{Proj}(S)}(-1))$, the *canonical bundle on* $\mathrm{Proj}(S)$. You may be able to understand these notations at the end of the lecture.

$\square$

Remark. Some interpretation. Say $S_0 = k$. Take generators in degree 1 of S as a k-algebra. It gives a closed embedding in $\mathbb{P}^n_k$. We can therefore interpret X as a certain subset of lines in k^{n+1}. The *cone* consists of those lines in $\mathbb{A}^{n+1}_k$. The vertex v correspond to the origin where all lines meet. Recall that X can be seen as the $\mathbb{G}_m$-quotient of $\mathrm{Spec}(S) \setminus v$. Blowing up at the vertex replaces v by all directions going into v inside $\mathrm{Spec}(S)$, *i.e.* X. We will soon see that T is a *line bundle* over X, in fact the one associated to $\mathcal{O}_X(-1)$. The zero section of this line bundle correspond to the exceptional divisor in the above point of view.

Exercise 3. *Functions on integral schemes and S_2 property.* Let $X = \mathrm{Spec}(A)$ be an integral affine scheme. Denote by η the generic point of X. Denote by $K(X) = \mathcal{O}_{X,\eta}$. Let $f \in K(X)$. Define

$$I = \{ g \in A \mid fg \in A \},$$

the *ideals of denominators* of f.

(1) Show that $X \setminus V(I)$ is the largest open subscheme U of X such that $f \in \mathcal{O}_X(U)$.
(2) *Geometric interpretation of the S_2 property.* Suppose that X is an affine integral S_2 scheme. Let $f \in K(X)$. Show that if $V(I)$ has codimension[3] at least 2, then $V(I)$ is empty.
(3) Deduce that if X is an affine integral S_2 scheme then if $f \in \mathcal{O}_X(U)$ with $X \setminus U$ being of codimension at least 2, then $f \in \mathcal{O}_X(X)$.

---

[3] meaning that every irreducible component of $V(I)$ has codimension at least 2

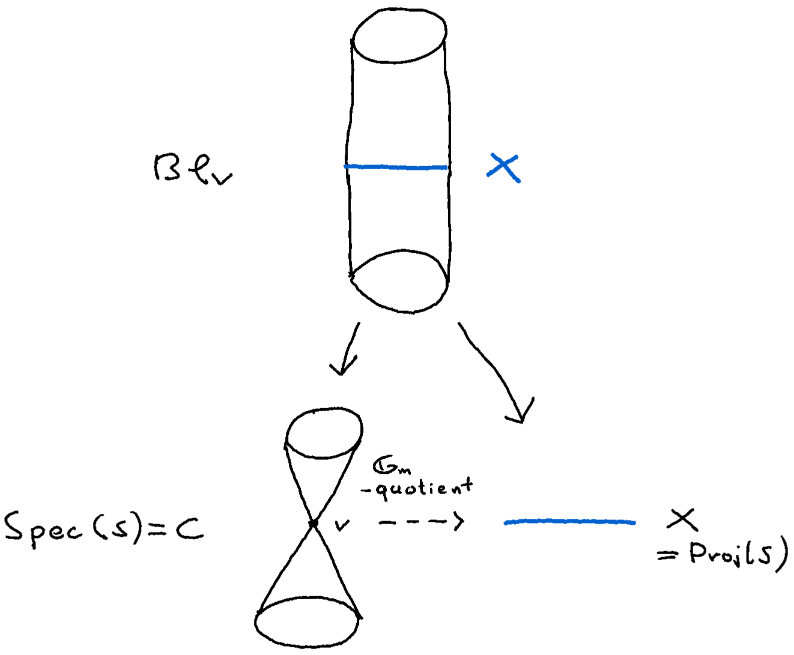

FIGURE 1. Some illustration of the above remark about Exercise 1.

Remark. We can drop the affine condition and do basically the same exercise using the language of quasi-coherent sheaves.

Solution key.

(2) Denote by A the integral domain we are now working with, let $f \in K = \mathrm{Frac}(A)$ and denote by I the ideal of denominators. Let $\mathfrak{p} \in V(I)$ a minimal prime. Therefore $\sqrt{IA_{\mathfrak{p}}} = \mathfrak{p}A_{\mathfrak{p}}$. By contradiction, suppose that it is of height at least 2. By the (S2) hypothesis, we see that therefore is a regular sequence $g, h \in \mathfrak{p}A_{\mathfrak{p}}$, and therefore without loss of generality in $IA_{\mathfrak{p}}$. Let also $a, b \in A$ such that

$$f = \frac{a}{g} = \frac{b}{h}.$$

This implies $ha = bh$. As (g, h) is regular, this implies that $a \in (g)$ reducing $\mod (g)$. But then $f \in A_{\mathfrak{p}}$, implying that $1 \in \mathfrak{p}A_{\mathfrak{p}}$, a contradiction.

(3) If $f \in \mathcal{O}_X(U)$ then $U \subset X \setminus V(I)$. So $V(I) \subset X \setminus U$. By the previous point, $V(I)$ is empty.

□

Exercise 4. *Fibers (2).*

(1) Show that for the morphism $\mathrm{Spec}(k[x, y]/(xy)) \to \mathrm{Spec}(k[x])$, induced by the obvious map $k[x] \to k[x, y]/(xy)$, every fiber is irreducible, although the $\mathrm{Spec}(k[x, y]/(xy))$ is not.

(2) Show that for the morphism $\mathrm{Spec}(\mathbb{Q}[t]) \to \mathrm{Spec}(\mathbb{Q}[t])$ induced by $t \mapsto t^2$ there are infinitely many closed points with irreducible fibers and infinitely many closed points with non-irreducible fibers.

Exercise 5. *Separated schemes.* Use the definition of separated maps to show the following.

(1) Show that a scheme X is separated if and only for every pair of open affines U and V, the intersection $U \cap V$ is affine and the natural map $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective. Show also that this holds if we suppose that there exists a cover (U_i) by open affines such that all their intersections (U_{ij}) are affines with $\mathcal{O}_X(U_i) \otimes \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_{ij})$ is surjective.
(2) Show that $\mathrm{Proj}(B) \to \mathrm{Spec}(\mathbb{Z})$ is separated for any $\mathbb{N}$-graded ring B. (Using (1) can be handy)
(3) Let k be a field. Let X be the scheme which is the gluing of two copies of $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ on the open subscheme $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. Show that $X \to \mathrm{Spec}(k)$ is not separated.

Solution key. (1) We show that a scheme X is separated if and only if for every (for a cover by) affine opens U and V (or for all pairs of any affine cover) of X

- $U \cap V$ is again affine,
- the natural map $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.

If X is separated note that $U \cap V$ can be realized as the *closed* subscheme of the affine scheme $U \times V = \mathrm{Spec}(\mathcal{O}(U)) \times \mathrm{Spec}(\mathcal{O}(V))$ which is $(U \times V) \cap \Delta$, where $\Delta \subset X \times X$ is the diagonal and is a closed subscheme by very assumption. This explains that the conditions are necessary. To see that they are sufficient, note that to check that $\Delta \subset X \times X$ is a closed immersion we can check it locally on *any* affine cover $(U_i \times U_j)_{ij}$ of $X \times X$ where $(U_i)_i$ is any affine cover of X. But now we see that the first condition ensures that the intersection $\Delta \cap U_i \times U_j$ is affine, and the second condition tells that it is a closed immersion.

(2) Now for a scheme of the form $\mathrm{Proj}(B)$, it suffices to show it for standard opens of the form $D_+(f)$ and $D_+(g)$ for f and g homogeneous. The intersection is affine, being $D_+(gf)$. We need to show that

$$B_{(f)} \otimes B_{(g)} \to B_{(fg)}$$

is surjective. But any degree zero element $\frac{h}{f^i g^j}$ can be written as a product of degree zero elements $\frac{bh}{f^N} \frac{f^N}{g^M}$ for appropriate $b \in B$ and $N, M \in \mathbb{N}$.

(3) We can use the criterion displayed above. Denote by $\mathbb{A}^1_a$ and $\mathbb{A}^1_b$ the two different copies of $\mathbb{A}^1$ inside this scheme. Their intersection is their common $\mathbb{G}_m$ by construction. It is the second property in the criterion above that fails. Namely

$$k[t] \otimes k[t] \to k[t, t^{-1}]$$

is not surjective, missing t^{-1}. This shows that $X \to \operatorname{Spec}(\mathbb{Z})$ is not separated. As a byproduct $X \to \operatorname{Spec}(k)$ is also not separated. Indeed if it was, as $\operatorname{Spec}(k) \to \operatorname{Spec}(\mathbb{Z})$ is, $X \to \operatorname{Spec}(\mathbb{Z})$ would also. ☐

Exercise 6. *Gradings, geometrically.* In what follows we write

$$\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}]).$$

Note that $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ defined by $t \mapsto xy$ gives a map of schemes[4]

$$m\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$$

which we call the *multiplication.*

A $\mathbb{G}_m$-action on an affine scheme $X = \operatorname{Spec}(A)$ is a map[5]

$$\mu\colon \mathbb{G}_m \times X \to X$$

such that the following diagram commutes[6]

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \times X \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "m \times \mathrm{id}"'] & \mathbb{G}_m \times X \arrow[d, "\mu"] \\
\mathbb{G}_m \times X \arrow[r, "\mu"] & X
\end{tikzcd}$$

and if c denotes the map $c\colon X \to \mathbb{G}_m \times X$ induced by the evaluation a 1, then $\mu c = \mathrm{id}$.

(1) Show that the set of $\mathbb{Z}$-gradings on a ring A (meaning gradings that turn A into a graded ring) is in one to one correspondence with the set of $\mathbb{G}_m$-actions on $\operatorname{Spec}(A)$.

(2) Let $d > 1$, S and R two $\mathbb{Z}$-graded rings, with associated action μ_S and μ_R. Show that (where $(-)^d\colon \mathbb{G}_m \to \mathbb{G}_m$ is induced by Spec by the ring map determined by $t \mapsto t^d$)

$$\begin{tikzcd}
\mathbb{G}_m \times \operatorname{Spec}(S) \arrow[r, "\mu_S"] \arrow[d, "((-)^d{,}f)"'] & \operatorname{Spec}(S) \arrow[d, "f"] \\
\mathbb{G}_m \times \operatorname{Spec}(R) \arrow[r, "\mu_R"] & \operatorname{Spec}(R)
\end{tikzcd}$$

commutes[7] if and only φ factors through $S^{(d)} = \bigoplus_n S_{nd}$, meaning that is it *homogeneous of degree d.* Compare with the notion introduced in Exercise 5, week 4.

(3) Let A be a $\mathbb{Z}$-graded ring and $A_0 \to A$ the inclusion. Show that $\pi\colon \operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ has the following universal property. For every affine scheme X and map $f\colon \operatorname{Spec}(A) \to X$ with the property

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[4] We have $\mathbb{G}_m \times \mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[x, x^{-1}, y, y^{-1}]$.

[5] We have $\mathbb{G}_m \times X = \operatorname{Spec}(A[t, t^{-1}]$.

[6] We have $\mathbb{G}_m \times \mathbb{G}_m \times X = \operatorname{Spec}(A[x, y, x^{-1}, y^{-1}]$.

[7] On points, this means that $f(\lambda x) = \lambda^d f(x)$.

that

$$\begin{tikzcd}
\mathbb{G}_m \times \operatorname{Spec}(A) \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] & \operatorname{Spec}(A) \arrow[d, "f"] & (\mathbb{G}_m\text{-invariant maps}) \\
\operatorname{Spec}(A) \arrow[r, "f"] & X &
\end{tikzcd}$$

then there exists a unique map $\overline{f}\colon \operatorname{Spec}(A_0) \to X$ with $\overline{f}\pi = f$. It means by definition that $\pi : \operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ is the quotient by the action of $\mathbb{G}_m$ in the category of affine schemes.

Solution key. This exercise was to hand in in a previous years so solutions are credited to past students who wrote them.

(1)(Daniil) Let's use the equivalence of categories of rings and affine schemes. So to each diagram from the definition of the action of $\mathbb{G}_m$ on an affine scheme X corresponds one to one with a map of rings $\mu^*\colon A \to A[t, t^{-1}]$ such that

$$\begin{tikzcd}
A[x, x^{-1}, y, y^{-1}] & A[t, t^{-1}] \arrow[l, "\mathrm{id}\otimes\mu^*"'] \\
A[t, t^{-1}] \arrow[u, "(t\to xy)\otimes\mathrm{id}"] & A \arrow[l, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}$$

and the composition of μ with evaluation at 1 (denoted by c) is id.
Write $\mu(a) = \sum\limits_i a_i t^i$. The commutativity of the diagram means

$$\sum_i a_i (yx)^i = ((t \to xy) \otimes \mathrm{id})\,(\sum_i a_i t^i) = ((t \to xy) \otimes \mathrm{id})\,(\mu^*(a))$$

$$\overset{\text{commutativity of the diagram}}{=} (\mathrm{id} \otimes \mu^*)\,(\mu^*(a)) = (\mathrm{id} \otimes \mu^*)\,(\sum_{i=1}^{n} a_i t^i) = \sum_i \mu^*(a_i) x^i.$$

In other words, the diagram above commutes if and only if $\mu^*(a_i) = a_i y^i$ *i.e.* if and only if a_i belongs to $A_i = (\mu^*)^{-1}(At^i)$. As for the second condition, it holds if and only if $a = \sum_i a_i$.
It now follows that $A = \bigoplus A_i$. The sum is direct because μ is an injective map of abelian groups.
Note also that $A_i A_j \subset A_{i+j}$ because μ^* is a morphism of rings. We then see that this gives an associated grading on the ring A.
Now given a grading on $A = \oplus A_i$ where we denote $a = \sum a_i$ we see that $\mu^*\colon A \to A[t, t^{-1}]$ sending $a \mapsto \sum a_i t^i$ is a ring morphism and using translations in algebra of the two conditions of a $\mathbb{G}_m$-action, we see that μ^* is indeed one. These constructions are by construction inverse to each other.

(3)(Karl) Since X is an affine scheme, let's just say $X = \operatorname{Spec}(B)$. The morphism f correspond to some morphism $\phi : B \to A$. Let ι denote the inclusion $A_0 \to A$. The pr_2 is induced by the inclusion of A into $A[t, t^{-1}]$, which we'll call i. We want to show that there is a unique $\bar{\phi}$ such that the diagram below commutes.

$$\begin{tikzcd}
A[t,t^{-1}] & & A \arrow[ll, "\mu^*"'] \\
& A_0 \arrow[ur, "\iota"] \arrow[dl, "\iota"'] & \\
A \arrow[uu, "i"] & & B \arrow[ll, "\phi"'] \arrow[uu, "\phi"'] \arrow[ul, dashed, "\bar{\phi}"]
\end{tikzcd}$$

We note that for $b \in B$, $i(\phi(b)) = \mu^*(\phi(b))$ if and only if $\phi(b) \in A_0$. So we can define $\bar{\phi} = \phi^{|A_0}$ which is necessarily unique because ι is injective.

$\square$

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AG II - Schemes and sheaves

Solutions – week 8

Exercise 1. *Closed subschemes.*

(1) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf. Let $Z = \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$. Denote by $\iota : Z \to X$ the inclusion. Show that $(Z, \iota^*\mathcal{O}_X/\mathcal{I})$ is a scheme. In what follows, $V(\mathcal{I})$ denotes the above associated scheme.
Hint: This is a local question. So using that $\mathcal{I}$ is quasi-coherent you can reduce to the case where X is affine and $\mathcal{I}$ correspond to an ideal.

(2) Show that $V(\mathcal{I})$ is a closed subscheme of X.

(3) Show that

$$\{\text{Quasi-coherent ideals of } \mathcal{O}_X\} \longleftrightarrow \{\text{Closed subschemes of } X\}$$

sending $\mathcal{I}$ to $V(\mathcal{I})$ is a one-to-one correspondence.

(4) Let $\operatorname{Spec}(A)$ be an affine scheme. Show that

$$\{\text{Ideals of } A\} \longleftrightarrow \{\text{Closed subschemes of } \operatorname{Spec}(A)\}$$

sending $I \mapsto (\operatorname{Spec}(A/I) \to \operatorname{Spec}(A))$ is a one-to-one correspondence.
Hint: You may use the equivalence of categories between quasi-coherent sheaves of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules and A-modules. For a proof which does not use this fact, see the solution of exercise 5, week 6.

Exercise 2. *Intersection of affine schemes.* Let X be a scheme and $U, V \subset X$ be open affine sub-schemes.

(1) Show that if X is separated then $U \cap V$ is affine.
Hint: Show that $U \cap V \cong X \times_{X\times X} (U \times V)$.

(2) Show that $U \cap V$ is not necessarily affine if X is not separated.
Hint: remember this open of an affine which is not affine? Play with this.

Solution key. For the first point, the claim follows from the Hint because the intersection is realized has a closed subscheme of an affine scheme. For the second point, one can take the affine plane with two origins. □

Exercise 3. *A map from a proper scheme to a separated scheme is closed.* Let $f\colon X \to Y$ be a map of S-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph $(\mathrm{id}, f) = \Gamma_f \colon X \to X \times_S Y$ is a closed immersion.

(2) Let $Z \subset X$ a closed subscheme proper over S. Show that $f_{|Z}$ is closed.

Solution key. The first point follows because

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\ X \times_S Y \arrow[r, "f\times \mathrm{id}"] & Y \times_S Y \end{tikzcd}$$

is a pullback square. The second claim follows because $Z \to Z \times_S Y \to Y$ is closed, the first map being closed by the first point and the second map being closed by universal closedness of $Z \to S$.

□

Remark. This fact is analogue to the topological result that a continuous map from a compact topological space to a Hausdorff space is always closed.

Exercise 4. *Morphisms into separated schemes.* Let S be a scheme. Let $X \to S$ and $Y \to S$ be S-schemes. Suppose that X is reduced and $Y \to S$ separated. Show that two morphisms of S-schemes

$$f_1, f_2 \colon X \to Y$$

that coincide an open dense subset of X are equal.
Give counter-examples if one of the hypotheses is dropped.

Solution key. Let Z be the scheme where $f_1 = f_2$ *i.e.* the pullback

$$\begin{tikzcd} Z \arrow[r, "f_1=f_2"] \arrow[d] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "f_1\times f_2"] & Y \times_S Y \end{tikzcd}$$

Because Y is separated Z is closed in X. Because of the assumption, $Z = X$ topologically, but then schematically because X is reduced.
We provide a counter-example if X is not reduced. Consider the two k-algebras maps (k being a field say)

$$k[x] \mapsto k[x,y]/(xy, y^2)$$

sending x to x and $x + y$ respectively. The induced maps on Spec agree on $D(x)$ which is dense. □

Remark. This fact is analogue to the topological result that if two continuous morphisms to a Hausdorff space agree on a open dense then they actually agree everywhere.

Exercise 5. *Generically finite morphisms.*

(1) Let k be a field. If $k \to A$ is finite, show that every prime of A is maximal.

Let $f \colon X \to Y$ be a dominant morphism between integral schemes.

(2) If f is finite, show that $\dim(X) = \dim(Y)$.
Hint: reduce to the affine case. Then use going up and that the map is surjective. Use point (1) to deduce that if $A \to B$ is finite and the preimage of two primes in B is the same in A, then the two primes are not included in one another.

(3) If f is finite type and $K(Y) \subset K(X)$ is a finite extension of fields, show that there exists an non-empty open $U \subset Y$ such that $f\colon f^{-1}(U) \to U$ is a finite morphism.
Hint: first prove the case where both X and Y are affine and then battle to use this case to conclude.

Solution key. (1) Because A is a finite dimensional k-vector space, it is Noetherian and Artinian. Let J be the Jacobson radical of A. Because of the Artinian hypothesis, $J^n = J^{n+1}$ for some n. Then by Nakayama $J^n = 0$. It implies that the Jacobson radical is equal to the radical.

Also, A has a finite number of maximal ideals. Indeed if not say $(\mathfrak{m}_i)$ is an infinite list of distinct maximal ideals. Then by the Artinian property, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{r+1}$. But then $\mathfrak{m}_1 \cdots \mathfrak{m}_r \subset \mathfrak{m}_{r+1}$, a contradiction (see a similar argument below). Say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals.

If $\mathfrak{p}$ is prime, then $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r \subset J \subset \mathfrak{p}$. If for all $\mathfrak{m}_i$ we have $\mathfrak{m}_i \not\subseteq \mathfrak{p}$, then we have elements $x_i \in \mathfrak{m}_i \notin \mathfrak{p}$. But then $x_1 \cdots x_r \in \mathfrak{p}$ a contradiction.

(2) The dimension is equal to the one of a dense open. So we can reduce to the affine case. Now, let $A \to B$ be finite between integral domains. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ a maximal chain of primes in A, meaning that there is no prime lying in between those. Because the map is finite, the map on Spec is surjective so we have $\mathfrak{q}_0$ lying over $\mathfrak{p}_0$. By going up we can lift to a chain in B. We now argue that we can not fit any more primes in the above list. If so, we would have two primes $\mathfrak{q} \subset \mathfrak{q}'$ with the same image in $\mathrm{Spec}(A)$. But by the previous point if two primes are in the same fiber, because the fiber has only maximal ideals, we see that $\mathfrak{q} = \mathfrak{q}'$.

(3) Note that without loss of generality Y is affine. We first begin by supposing that X is also affine. We are then in the situation of a finite type map $A \to B$ between integral domains, such that this map induces a finite extension of fields at the fields of fractions. Say that $b_1, \ldots, b_n$ generates B as an A-algebra. By hypothesis, there exists polynomials $f_i \in \mathrm{Frac}(A)[t]$ such that $f_i(b_i) = 0$. Therefore there exist a non-zero element $g \in A$, namely the product of the denominators of each coefficients of the polynomials f_i, such that $b_1, \ldots, b_n$ are integral over A_g. This implies that $A_g \to B_g$ is finite.

Now we show the general case. As $f\colon X \to Y$ is finite type over an affine scheme, there exists a finite covering by affine schemes X_i of X. By the preceding case, there exists $U_i \subset Y$ such that $f^{-1}(U_i) \cap X_i \to U_i$ is finite. We may replace Y by the intersection of the U_i's and also suppose that it is affine. With this reduction, we are now in the following situation: we have a covering X_i of X

such that $X_i \to Y$ is finite. Let V be the intersection of the X_i's. Say that $A \to B$ is the finite ring map corresponding to $X_1 \to Y$. Say that $0 \neq b \in B$ is such that $D(b) \subset V$. Note that as b is integral over A, there is a polynomnial with non zero-constant coefficient $\sum_{i=0}^{n} a_i t^i \in A[t]$ such that $f(b) = 0$. Therefore,

$$b \left(\sum_{i=1}^{n} a_i b^{i-1} \right) = -a_0$$

Therefore, there is a non-zero element $b' \in B$ such that $a := bb' \in A$. It implies that $f^{-1}(D(a)) \subset D(b) \subset V \subset X_1$. Therefore $f \colon f^{-1}(D(a)) \to D(a)$ is finite.

□

Exercises 6, 7, and 8 are purely about the underlying topology of the schemes in question.

Exercise 6. *Projection from affine spaces.* Let R be a ring.

(1) Show that

$$\pi \colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$

is open. More precisely, if $f = \sum a_i t^i$ show that

$$\pi \left(D(f(t)) \right) = \bigcup_i D(a_i).$$

(2) Let $g(t) \in R[t]$ be a monic polynomial and $f(t) \in R[t]$. Remark that $R[t]/g(t)$ is a free R-module of rank $\deg(g)$. Let $\chi(X) = \sum_i^n r_i X^i$ be the characteristic polynomial of the multiplication by $f(t)$ on $R[t]/g(t)$. Show that

$$\pi(D(f) \cap V(g)) = \bigcup_i^{n-1} D(r_i).$$

Solution key. (1) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\mathfrak{p} \in \pi(D(f(t)))$ if and only if $k(\mathfrak{p})[t]_{f(t)} \neq 0$ if and only if $f(t) \neq 0$ in $k(\mathfrak{p})[t]$ if and only if there is some i such that $a_i \notin \mathfrak{p}$.

(2) Let $\mathfrak{p} = \mathfrak{q} \cap R$ with $\mathfrak{q} \in D(f) \cap V(g)$ in the image. So we have a map $k(\mathfrak{p})[t]/(g(t)) \to k(\mathfrak{q})$. Note the following fact: by Cayley-Hamilton f is nilpotent in $k(\mathfrak{p})[t]/(g(t))$ if and only if $\mathfrak{p} \in V(r_0, \dots, r_{n-1})$.

Note also that $f \neq 0$ in $k(\mathfrak{q})$ because $\mathfrak{q} \in D(f)$. So f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$ and therefore $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$.

Reciprocally if $\mathfrak{p} \in \bigcup_{i=1}^{n1} D(r_i)$, the by the above argument f is not nilpotent in $k(\mathfrak{p})[t]/(g(t))$. Therefore there is some $\mathfrak{q} \not\ni f$ in $k(\mathfrak{p})[t]/(g(t))$ meaning that $\mathfrak{q} \in D(f) \cap V(g)$, which is therefore sent to $\mathfrak{p}$.

□

Exercise 7. *Chevalley's theorem.* Let X be a Noetherian topological space. A subset $T \subset X$ is called *constructible* if it can be written as a finite union of sets of the form $U \cap V^c$ where U and V are open sets.

(1) Show that if $X = \mathrm{Spec}(R)$ for a Noetherian ring R, a subset is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \dots, g_m)$ with $f, g_1, \dots, g_m \in R$.

(2) Show using exercise 6 that

$$\pi\colon \mathrm{Spec}(R[t]) \to \mathrm{Spec}(R)$$

sends constructible subsets to construtible subsets.
Hint: Show by induction on $\sum_i \deg(g_i)$ *that if* $f, g_1, \dots, g_m \in R[t]$ *are polynomials, the image of* $D(f) \cap V(g_1, \dots, g_m)$ *is constructible. To conduct the induction step, consider* α *the leading coefficient of* g_1*. Break down the study on the open and closed* $D(\alpha)$ *and* $V(\alpha)$ *to reduce the sum of the degrees.*

(3) Deduce *Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type morphism between Noetherian schemes. Then f sends constructible subsets to constructible subsets.[1]

Solution key. (2) Note that we already know two cases. Namely the case $D(f)$ and the case $D(f) \cap V(g)$ where g is monic. We proceed by induction on the sum of the degrees of g_i – also we order them such that they have increasing degrees. Let c be the dominant coefficient of g_1. We have

$$\mathrm{Spec}(R[t]) = \mathrm{Spec}(R/c[t]) \sqcup \mathrm{Spec}(R_c[t]).$$

In the first the image of g is of degree striclty less. So induction goes.

Also, note that g_1 is monic in $\mathrm{Spec}(R_c[t])$. If $n = 1$, we are in an already dealt situation. If not let

$$g_2' = g_2 - t^{d_2 - d_1}(c'/c)g_1$$

where c' is the leading coefficient of g_1. Then

$$D(f) \cap V(g_1, g_2, \dots, g_n) = D(f) \cap V(g_1, g_2', \dots, g_n).$$

But now the sum of degrees of the list lowers giving the claim by induction.

(3) We can reduce to the affine case where we can reduce to

$$R \to R[t_1, \dots, t_n] \to S$$

where the last map is surjective. The first arow induces on Spec a map which preserves constructibility by the above and the second also because it is a closed immersion

□

Remark. In general the topological image of a morphism of schemes can fail to be open or closed but in cases where Chevalley's theorem applies, it

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[1] The generalization to non-Noetherian settings requires more careful definitions, but once these definitions are addressed the proof is the same.

tells that it still not too far from it and manageable. In particular one can endow the image with a scheme structure.

Exercise 8. *An application of Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type dominant map between Noetherian schemes with Y irreducible. Use Chevalley's theorem to show that the topological image $f(X)$ contains an open set.

Solution key. The image being dense contains the generic point of Y, therefore, $\eta_Y \in U \cap Z \subset f(X)$ because the topological image $f(X)$ is constructible, for some open U and closed Z of Y. But if $\eta_Y \in Z$ then we see that $Z = Y$ $\square$

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AG II - Schemes and sheaves

Solutions – week 9

Exercise 1. *Nullstellensatz* via *Chevalley.*

(1) Let $\mathfrak{m}$ be maximal in $k[x_1, \dots, x_n]$. Suppose by contradiction that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}$[1] is not maximal, because it is prime, we have $\mathfrak{p}_i = (0)$. The image of the map $\operatorname{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is $\mathfrak{p}_i$. But by Chevalley, the image of the map $\operatorname{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is constructible, but by our hypothesis, also contains the generic point, and therefore contains an open set. But an open in $\mathbb{A}^1_k$ contains infinitely many points, way much than our singleton $\{\mathfrak{p}_i\}$, leading to a contradiction.

(2) We see by successive quotients, because each $\mathfrak{p}_i$ is maximal in $k[x_i]$, that $(\mathfrak{p}_1, \dots, \mathfrak{p}_n)$ is maximal. But has it is contained in $\mathfrak{m}$ we have our claimed equality. Also if we denote by $k[x_i]/(\mathfrak{p}_i) = k(\alpha_i)$ where α_i is therefore an algebraic element over k. Then

$$k[x_1, \dots, x_n]/\mathfrak{m} = k(\alpha_1, \dots, \alpha_n)$$

and therefore a finite extension.

(3) First, note that from the last point, we deduce that every residue field of a finite type k-algebra at a closed point is a finite extension of k. Let $\mathfrak{m}$ be maximal in $\operatorname{Spec}(B)$. Then we have injections

$$k \to A/f(\mathfrak{m}) \to B/\mathfrak{m}.$$

Because $B/\mathfrak{m}$ is finite dimensional over k, so is $A/f(\mathfrak{m})$. But then the multiplication by every non-zero element is injective, but then surjective because it is a self of a finite dimensional k-vector space. We conclude that $A/f(\mathfrak{m})$ is a field, leading to the desired conclusion.

(4) It suffices to show that every element which is not nilpotent is contained in some maximal ideal. If f is not nilpotent, then $A_f \neq 0$. So there is a maximal ideal in A_f. By the previous point, the preimage of this ideal is maximal in A, concluding.

Exercise 2. *Dual.*

For (2) and (3), the key is to consider the *natural maps* in the sense that for any map $\mathcal{E} \to \mathcal{E}'$ we have commuting diagrams

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}^{\vee\vee} \arrow[d] \\
\mathcal{E}' \arrow[r] & \mathcal{E}'^{\vee\vee}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}'^{\vee} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) \arrow[d] \\
\mathcal{E}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})
\end{tikzcd}$$

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[1] This is the projection to the i-th coordinate.

To show that these natural horizontal maps are isomorphisms, we can prove that the map is an isomorphism locally. But then, locally these shaves are isomorphic to a finite sum of $\mathcal{O}$, where for those the statement follows from standard linear algebra. Using the above squares, we get the general claim.

Exercise 3. *Compatibilities between f^*, f_* and $\otimes$.*
For (1), one may show first that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes, \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})).$$

Then the claim follows by combining adjunctions.
For (2), we proceed as in the previous exercise, meaning we construct a natural morphism between the implicit functors in $\mathcal{E}$, and then using this naturality we are allowed to show the claim locally. The natural map correspond by adjunction to tensoring the counit map $f^*\mathcal{E} \otimes (f^*f_*\mathcal{F} \to \mathcal{F})$.

Exercise 4. *Fibre dimension (of coherent sheaves).*
Because each question is local, say $X = \mathrm{Spec}(A)$, where A is Noetherian and we work with M global sections of $\mathcal{F}$, which is a finitely generated A-module. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. For (1), note that if $M(\mathfrak{p})$ is of dimension n, say with basis $m_1, \ldots, m_m$, then we have a surjective map by Nakayama

$$A_{\mathfrak{p}}^n \to M_{\mathfrak{p}}.$$

Find some $a \in A$ such that this surjections lifts to a map

$$A_a^n \to M_a.$$

The coker K of this map is finitely generated and satsfies $K_{\mathfrak{p}} = 0$. Therefore we may localize further to have $K_b = 0$ for some $b \in A$ and concluding that we have a surjective map

$$A_b^n \to M_b.$$

This implies that complements of sets in the statement are open.
For (2), note that φ is continuous to the discrete topology on $\mathbb{N}$ if $\mathcal{F}$ is locally free. Therefore only one fiber can be non-empty because fibers are disjoint opens and the union of all fibers cover the space.
As for (3), proceed as in (1) to get a surjective map

$$A_b^n \to M_b.$$

An element in the kernel is a vector $(a_1, \ldots, a_n)$ where each element is in every prime ideal of A_b. Indeed, for any prime ideal $\mathfrak{p}$ of A_b, looking at

$$k(\mathfrak{p})^n \to M(\mathfrak{p})$$

we have a surjective map between to $k(\mathfrak{p})$-vector spaces of the same dimension so also injective. Because A_b is reduced, the intersection of all primes is the zero ideal, concluding.

Exercise 5. *Fibre dimension (of finite type morphisms)* We recall some results along the way that you can assume.

Lemma 1 (Krull's height theorem). *Let R be a Noetherian ring. Suppose that $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_n)$. Then*

$$\mathrm{ht}(\mathfrak{p}) \leq n.$$

(1) Let R be a Noetherian ring and $\mathfrak{p}$ be a prime ideal. Using Krull's height theorem, show by induction on the height that for every prime $\mathfrak{p}$ of height n there is $(f_1, \dots, f_n) \subset \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_n)$ and every minimal prime of $(f_1, \dots, f_n)$ has height n.

(2) Let $f \colon X \to Y$ be a morphism between locally Noetherian schemes and $Y' \subset Y$ a closed irreducible subset. Show that for every irreducible component $Z \subset f^{-1}(Y')$ that dominates Y' we have

$$\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y).$$

Hint: This is a local problem so you can reduce to affines and use item (1).

Lemma 2. *Let k be a field, A be a finite type k-algebra which is also a domain and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then*

$$\dim(A) = \mathrm{trdeg}_k(\mathrm{Frac}(A))$$
$$\text{and } \mathrm{codim}(\mathrm{Spec}(A/\mathfrak{p}), \mathrm{Spec}(A)) = \mathrm{ht}(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p}).$$

(3) Let $f \colon X \to Y$ be a map between finite type integral k-schemes. Show that for every $y \in f(X)$ and Z irreducible component of X_y we have

$$\dim(X) - \dim(Y) \leq \dim(Z) \leq \dim(X).$$

Hint: Use item (2) with $Y' = \overline{\{y\}}$. Use lemma 2 and the additivity of transcendence degree with $k \mid k(y) \mid K(Z)$. Namely

$$\mathrm{trdeg}_k(K(Z)) = \mathrm{trdeg}_k(k(y)) + \mathrm{trdeg}_{k(y)}(K(Z)).$$

(4) Let $f \colon X \to Y$ be a dominant map between finite type integral k-schemes. Show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(X_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open. *Hint: show that you can reduce to the affine case* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *with $t_1, \dots, t_e \in B$, where $e = \dim(X) - \dim(Y)$, such that $t_1, \dots, t_e$ form a transcendence basis of $K(X)$ over $K(Y)$. Then factor the morphism by* $\mathrm{Spec}(A[t_1, \dots, t_n])$. *Note that $X \to \mathrm{Spec}(A[t_1, \dots, t_e])$ induces a finite morphism at fraction fields and that* $\mathrm{Spec}(A[t_1, \dots, t_e]) \to \mathrm{Spec}(A)$ *is isomorphic to $\mathbb{A}^e_A \to \mathrm{Spec}(A)$ which is open by exercise 2. Use exercise 1.(2) to conclude.*

Remark. You are free to prove the following weaker version of the statement: show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(U_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open.

(5) Let $f \colon X \to Y$ be a dominant map between finite type integral k-schemes. For $h \in \mathbb{N}$, let E_h be the set of points x of X such that

there is an irreducible component of $X_{f(x)}$ with dimension at least h, which contains x. Show that E_h is closed.[2]
Hints: If $h \leq e$, then $E_h = X$ by (3). If $h > e$, note that $E_h \subset X \setminus U$ where U is the open of item (4). Proceed by induction on the dimension of X.

(6) Let $f\colon X \to Y$ be a closed map between finite type integral k-schemes. For $h \in \mathbb{N}$, let F_h be the set of points of y of Y such that there is an irreducible component of X_y with dimension at least h. Show that F_h is closed.
Hint: Show that $F_h = f(E_h)$.

This exercise was hand in in a previous year and therefore solutions are attributed to students who wrote them.

(1)(Joel) Suppose we have proved the statement for $n = k$, and let $\mathfrak{p}$ be a prime of height $k+1$. Choose a prime $\mathfrak{q} \subset \mathfrak{p}$ of height k, so by induction there exist $\{f_1, \ldots, f_k\}$ such that $\mathfrak{q}$ is a minimal prime of $I = (f_1, \ldots, f_k)$ and $\mathrm{ht}(\mathfrak{q}) = k$. Let $\{q_i\}$ be the minimal primes of I (so $\mathrm{ht}(q_i) = k$ by induction). As $\mathrm{ht}(\mathfrak{q}_\mathrm{i}) < \mathrm{ht}(\mathfrak{p})$ for all i, $\mathfrak{p} \not\subseteq \mathfrak{q}_i$, so by prime avoidance $\mathfrak{p} \not\subseteq \cup \mathfrak{q}_i$. Hence, there exists some $f_{k+1} \in \mathfrak{p} \setminus \cup \mathfrak{q}_i$. Define $I' = (f_1, \ldots, f_{k+1})$, and let $\mathfrak{p}'$ be a minimal prime of I'. By Krull's height theorem $\mathrm{ht}(\mathfrak{p}') \leq k+1$. As $\mathfrak{q}_j \subsetneq \mathfrak{p}'$ for some $\mathfrak{q}_j$ (as by our choice of f_{k+1}), we have $\mathrm{ht}(\mathfrak{p}') > \mathrm{ht}(\mathfrak{q}_j)$, and hence $\mathrm{ht}(\mathfrak{p}') = k+1$ for any minimal prime $\mathfrak{p}'$ of I'. As $\mathfrak{p}$ contains all the generators of I' and is of height $k+1$ by assumption, $\mathfrak{p}$ is a minimal prime of I', as else we could fit a minimal prime $\mathfrak{q}'$ of height $k+1$ in $I' \subsetneq \mathfrak{q}' \subsetneq \mathfrak{p}$, contradicting $\mathrm{ht}(\mathfrak{p}) = k+1$.

(2)(Joel) As $\mathrm{codim}(Z, X) = \dim \mathcal{O}_{Z,\eta}$ for the generic point η of Z', and similarly for $Y' \subset Y$, the question is local and we can reduce to the case where $f : X = \mathrm{Spec}\ B \to Y = \mathrm{Spec}\ A$ and $\varphi : A \to B$ is the corresponding ring map. Let $\mathfrak{p} \in \mathrm{Spec}\ A$ be such that $V(\mathfrak{p}) = Y'$ and $Z \subset f^{-1}(\mathfrak{p})$ be an irreducible component of $f^{-1}(\mathfrak{p}) = \mathfrak{p}^e$, where $\mathfrak{p}^e$ denotes the extension of $\mathfrak{p}$ by φ. Suppose $\mathfrak{p}$ has height n, so by part (1) there exist $f_1, \ldots, f_n \in A$ such that $\mathfrak{p}$ is a minimal prime of $I = (f_1, \ldots, f_n)$. Then $\mathfrak{p}^e \supseteq I^e = (\varphi(f_1), \ldots, \varphi(f_n))$. Let $\mathfrak{q}$ be a minimal prime of $\mathfrak{p}^e$ corresponding to the irreducible component Z. Next, we show that $\mathfrak{q} \supseteq \mathfrak{p}^e$ is a minimal prime of I^e, which by Krull's height theorem implies that $\mathrm{ht}(\mathfrak{q}) \leq n = \mathrm{ht}(\mathfrak{p})$, which is equivalent to the inequality $\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y)$.
First, we may assume that $\mathfrak{p}^e \supsetneq I^e$, as if $\mathfrak{p}^e = I^e$, the result is immediate as $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}^e$. Now, suppose there exists $\mathfrak{r} \in \mathrm{Spec}\ B$ such that $\mathfrak{q} \supsetneq \mathfrak{r} \supseteq I^e$. Then $\varphi^{-1}(\mathfrak{q}) \supseteq \varphi^{-1}(\mathfrak{r}) \supseteq \varphi^{-1}(I^e) \supseteq I$. As Z dominates Y', $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and furthermore as $\mathfrak{p}$ is a minimal prime of I. As $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}^e = \varphi^{-1}(\mathfrak{r})^e \subseteq \mathfrak{r}$, and $\mathfrak{r}$ is by assumption prime, we see that $\mathfrak{q} = \mathfrak{r}$, and hence $\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y)$.

(3)(Joel) Again, the question is local, so let $f : X = \mathrm{Spec}\ B \to \mathrm{Spec}\ A = Y$ be a morphism of affine schemes, and $y \in Y$ be a point corresponding to

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[2] The statement is true for any $f\colon X \to Y$ between X and Y finite type k-schemes without the dominant hypothesis. This can be shown by an easy reduction to the case of the exercise.

$\mathfrak{p} \in \operatorname{Spec} A$. Set $Y' := \bar{\{y\}}$, the closure of y in Y. Let $\phi : A \to B$ be the ring map corresponding to f.

B is a finitely generated $A-$algebra, so $B \bigotimes_A k(y)$ is a finitely generated $k(y)$-algebra, and hence $\dim Z = \operatorname{trdeg}_{k(y)}(K(Z))$ for any any irreducible component Z of $f^{-1}(y)$, as Z is of finite type over $k(y)$ and the trace formula holds.

If Z is an irreducible component of X_y, we want to show that $\bar{Z}$ is an irreducible component of $f^{-1}(Y')$. Z is contained in some irreducible component W of $f^{-1}(Y')$. As $Z \subset W$ and $Z \subset X_y$, the image of W contains $y \in Y$. Let η be the generic point of W. Then $f(W) \subset Y'$ is a dense inclusion with both W and Y' irreducible, and so $f(\eta) = y$, the generic point of Y'. Note that the closure of η intersected with X_y is irreducible and contains Z, and hence $W = \bar{\{\eta\}} \subset \bar{Z}$, so $\eta \in Z$. By part (2) we have $\operatorname{codim}(\bar{Z}, X) \leq \operatorname{codim}(Y', Y)$. As Z is dense in $\bar{Z}$, $K(Z) = K(\bar{Z}$, and similarly for $K(y)$ and $K(Y')$. Using the trace formula for $k - k(y) - k(Z)$, we get $\operatorname{trdeg}_{K(y)} K(Z) = \operatorname{trdeg}_k K(Z) - \operatorname{trdeg}_k K(y) = \operatorname{trdeg}_k K(\bar{Z}) - \operatorname{trdeg}_k K(Y')$. Using the inequality for codimensions we get $\dim X - \dim Y \leq \dim \bar{Z} - \dim Y' = \operatorname{trdeg}_k K(\bar{Z}) - \operatorname{trdeg}_k K(Y') = \operatorname{trdeg}_{K(Y')} K(\bar{Z}) = \operatorname{trdeg}_{K(y)} K(Z) = \dim Z$. As $X_y \simeq f^{-1}(y) \subset X$, we get that $\dim Z \leq \dim X$, and hence in total we have

$$\dim X - \dim Y \leq \dim Z \leq \dim X.$$

(4)(Héloïse) We prove the following.

Lemma. *Let $f : X \to Y$ be a dominant map between finite type integral k-schemes. There is an open dense subset $V \subset X$ such that for all $y \in f(V)$,*

$$\dim(X_y) = \dim(X) - \dim(Y)$$

and $f(V)$ is open.

We begin by proving the following weaker statement.

Lemma. *Let $f : X \to Y$ be a dominant map between finite type integral k-schemes. There is an open dense set $U \subset X$ such that for all $y \in f(U)$*

$$\dim(U_y) = \dim(X) - \dim(Y).$$

Proof. Note that we are free to reduce Y to an affine dense open and also U can be taken to be inside a dense affine open of X, so we can reduce to the affine case, as we do in what follows.

Proof of the affine case. We denote by $\phi : A \to B$ the ring map corresponding to f. Note that since f is a morphism between k-schemes, ϕ is injective. Let $e := \dim(X) - \dim(Y)$.

By using the additivity of the trenscendance degree (since f is a dominant map between finite type integral k-schemes) to the field extensions $K(X) \mid K(Y) \mid k$ induced by ϕ, we get that

$$\begin{aligned} e &= \dim(X) - \dim(Y) \\ &= \operatorname{trdeg}_k(K(X)) - \operatorname{trdeg}_k(K(Y)) \\ &= \operatorname{trdeg}_{K(Y)}(K(X)). \end{aligned}$$

Let $\{t_1, \dots, t_e\}$ be a transcendence basis of $K(X)$ over $K(Y)$. Note that the elements $t_i \in K(X)$ may be seen as fractions with numerators in A and denominators in B therefore, by considering f to be the product of all the denominators of the t_i's (which is finite since $e < \infty$), and localizing A at f, we get that the t_i's are elements of A_f. From there, we get the following commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "\phi"] \arrow[dr, "i"'] & & A_f \\
& B[t_1, \dots, t_e] \arrow[ur, "j"'] &
\end{tikzcd}$$

which induces the following diagram on affine schemes.

$$\begin{tikzcd}
D(f) \arrow[rr, "f"] \arrow[dr, "\eta"'] & & \mathrm{Spec}(B) \\
& \mathbb{A}^e_B \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Since we are looking for a dense open subset, we sloppily rename $\mathrm{Spec}(A) := D(f)$ for the rest of the proof. By exercise 2 of sheet 8, the map $\iota : \mathbb{A}^e_B \cong \mathrm{Spec}(B[t_1, \dots, t_e]) \to \mathrm{Spec}(B)$ is open, while the map $\eta : \mathrm{Spec}(A) \to \mathbb{A}^e_B$ induced by j is dominant since j is injective. Morevoer, η is finite type since it is a map of finite type integral k-schemes. By noting that the field extension

$$K(Y)(t_1 \dots t_n) \subseteq K(X) = \mathrm{Frac}(A)$$

is finite, since $\mathrm{trdeg}_{K(Y)}(K(X)) = e \leq \infty$, we conclude by exercise 1.2 of sheet 8 that there exists a non-empty open set $V \subset \mathbb{A}^e_B$ such that the restriction of the morphism η to $\eta^{-1}(W)$ is finite and dominant since η is dominant. In particular, it is closed. Then, since the map is dominant and closed, it is surjective.

Moreover, since η is surjective, $\eta(\eta^{-1}(W)) = W$ and therefore, $f(\eta^{-1}(W)) = \iota \circ \eta(\eta^{-1}(W)) = \iota(W)$ which is open since ι is an open map. Finally, since A is an integral domain, it is in particular irreducible and we conclude that $\eta^{-1}(W)$ is dense.

Now for any $\mathfrak{p} \in f(\eta^{-1}(W)) = \iota(W)$, since finitness and surjectivity of a morphism is stable under base change, the morphism g coming from the following base change is finite and surjective.

(1)

$$\begin{tikzcd}
(\eta^{-1}(W))_{\mathfrak{p}} \arrow[r] \arrow[d, "g"'] & \eta^{-1}(W) \arrow[d, "\eta"] & \\
W \times_Y k(\mathfrak{p}) \arrow[r] \arrow[d] & W \arrow[d] & le \\
k(\mathfrak{p}) \arrow[r] & Y &
\end{tikzcd}$$

Therefore,

$$\dim(\eta^{-1}(W))_{\mathfrak{p}}) = \dim(W \times_Y k(\mathfrak{p})) \leq \dim(\mathbb{A}^e_{k(\mathfrak{p})}) = e.$$

Now note that $B[t_1, \ldots, t_n]$ is an integral domain, hence $\mathbb{A}^e_B$ is an integral scheme. Therefore, $\dim(W) = \dim(\mathbb{A}^e_B) = \dim(B) + e$. Furthermore, since the restriction of the morphism η to $\eta^{-1}(W)$ is dominant, hence finite and surjective, $\dim(\eta^{-1}(W)) = \dim(W)$.

By applying the result from question 3 to the restriction of f to $\eta^{-1}(W)$, then for any irreducible component Z of the fibre $(\eta^{-1}(W))_{\mathfrak{p}}$, we get

$$\dim(B) + e - \dim(B) \leq \dim(Z).$$

Since the above holds for any irreducible component, we conclude that $e \leq \dim((\eta^{-1}(W))_{\mathfrak{p}})$. Thus

$$\dim((\eta^{-1}(W))_{\mathfrak{p}}) = e$$

and we can pick $U = \eta^{-1}(W)$. □

This lemma does not yet allow to generalize to the the whole fiber X_y, as the equality $\dim(U_y) = \dim(X_y)$ might not hold for any open dense set U. We therefore need to further refine U using the following lemma.

Lemma. *Let $f : X \to Y$ be a map between finite type integral k-schemes. Then, there exists a dense open set $V \subseteq Y$ such that for all $y \in Y$, $U_y \subset X_y$ is dense.*

Proof. Reduction to the affine case. Up to shrinking Y, we may assume that Y is an affine scheme $Y := \operatorname{Spec}(A)$.

Now, suppose that we have proven the statement when X is an affine scheme. For a general scheme X, consider an affine open cover $X = \bigcup_i W_i$ where W_i are affine schemes. For each W_i, there exists an open dense set $V_i \subseteq Y$ such that for any $y \in V_i$, $(U \cap W_i)_y \subset (W_i)_y$ is dense. Consier $V := \bigcup_i V_i$. Then for any $y \in V$, $U_y \subset X_y$ is dense. Indeed, the fiber X_y is a glueing of the $(W_i)_y$'s while U_y is a glueing of the $(U \cap W_i)$'s, where each $(U \cap W_i)_y$ is dense in $(W_i)_y$.

Proof of the affine case. Suppose that $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a map between finite type integral k-schemes. Since the principal open sets form a basis for the topology on $\operatorname{Spec}(B)$, up to shrinking U, we may assume that

U is of the form $D(t)$ with $t \in B$.
Consider the short exact sequence

$$(2) \qquad 0 \longrightarrow B \xrightarrow{\cdot t} B \longrightarrow B/(t) \longrightarrow 0\ .$$

By the *Generic flatness theorem*, there exists a dense open set $V \subset \operatorname{Spec}(A)$ such that for any $y \in V$, the morphism $B \otimes_A k(y) \to B_t \otimes_A k(y)$ is injective because $B/(t)$ can be supposed to be flat on this open, which implies that the morphism $(D(t))_y \to X_y$ is dominant. □

Let V be as in the previous lemma. By considering $U' := U \cap f^{-1}(V)$, we have proven the general case.
(5) (Alissa) Let $f : X \to Y$ a dominant map between finite type integral k-schemes. For h a positive or 0 integer we define

$$E_{X,h} := \{x \in X | \exists Z \subseteq X_{f(x)} \text{ an irreducible component which contains } x \text{ s.t. } \dim Z \geqslant h\}$$

Show that $E_{X,h}$ is closed.

We see that if $h \leqslant \dim X - \dim Y$ then by part 3 we conclude that $E_{X,h} = X$, hence it is closed. Now if we consider the case $h > \dim X - \dim Y$ then we see that if take the U obtained in Part 4 then $E_{X,h} \subseteq X \setminus U$. We proceed by induction on the dimension of X to prove that $E_{X,h}$ is closed. (take the version of Part 4 with X_y and not only U_y)
Suppose that $\dim X = 0$. Then we see that $0 \leqslant \dim Z \leqslant \dim X_y = \dim f^{-1}(y) \leqslant \dim X = 0$. So $E_{X,h}$ is just $\emptyset$. For the induction step, suppose that the result is true for every X of dimension $d-1$ or less. Suppose that $\dim X = d$. Then if we consider $E_{X,h}$ we see that $E_{X,h} \subseteq X \setminus U$ which is closed. So now we can consider the decomposition of $X \setminus U$ in a union of irreducible closed subsets C_i. The latter will have dimension strictly smaller than X since they are irreducible in X which is itself irreducible. Since X is a fintie type k-scheme, we see that there must be only finitely many C_i's. We would like to show that $E_{X,h} = \bigcup_{i=1}^n E_{C_i,h}$. To do so, notice first that we can endow each C_i with a reduced scheme structure. Since it is irreducible, we get that C_i is integral. If we show the above equality, we would like to use induction since the C_i's have strictly lower dimension than d. However, to apply induction we have to be in the good conditions. So we need an integral image and a dominant map. Furthermore, we need C_i and the image to be finite type k-schemes. So let us consider the morphism $f|_{C_i} : C_i \to \overline{f(C_i)}$ where we endow $\overline{f(C_i)}$ with the reduced scheme structure. Since C_i is irreducible, then $f(C_i)$ is too and so is $\overline{f(C_i)}$. It is direct that the morphism is dominant. Since X and Y are finite type k-schemes, then C_i and $\overline{f(C_i)}$ are finite type k-schemes. As before, $f|_{C_i}$ is finite type since it is a morphism between finite type k-schemes.
First, let $x \in E_{C_i,h}$. We would like to show that $\bigcup_{i=1}^n E_{C_i,h} \subseteq E_{X,h}$. Remember that we have

$$C_{i,f(x)} \approx f^{-1}(f(x)) \cap C_i \subseteq f^{-1}(f(x)) \approx X_{f(x)}$$

Since the C_i's are closed, $f^{-1}(f(x)) \cap C_i$ is a closed subset of $f^{-1}(f(x))$. Hence an irreducible component of $C_{i,f(x)}$ containing x is also an irreducible closed subset of $f^{-1}(f(x))$ containing x.

Now for the other inclusion let $x \in E_{X,h}$. Then we notice that

$$X_{f(x)} \approx f^{-1}(f(x)) = \cup_{i=1}^{n} f^{-1}(f(x)) \cap C_i$$

We see that the irreducible components of $X_{f(x)}$ are the irreducible components of each $f^{-1}(f(x)) \cap C_i$. This is how we get $E_{X,h} \subseteq \bigcup_{i=1}^{n} E_{C_i,h}$.

□

(6) (Alissa) Let $f : X \to Y$ a closed map between finite type integral k-schemes. For $h \in \mathbb{N}$ we define

$$F_h := \{y \in Y | \exists Z \subseteq X_y \text{ an irreducible component s.t. } \dim Z \geqslant h\}$$

Show that F_h is closed.

To show this, we will rather prove that $f(E_h) = F_h$. Since f is closed and using Part 5, it follows immediately that F_h is closed.
We will show that $f(E_h) = F_h$ by showing each inclusion.

$\underline{f(E_h) \subseteq F_h}$: Let $y \in f(E_h)$. Then there exists $x \in E_h$ such that $f(x) = y$. This implies that there exists an irreducible component of $X_{f(x)} = X_y$ of dimension at least h. Hence $f(x) \in F_h$ by definition.

$\underline{F_h \subseteq f(E_h)}$: Let $y \in F_h$. If $y \notin f(X)$ we see that the fiber must be the empty set since $X_y \approx f^{-1}(y)$. Hence we see that $y \in f(X)$. We know that there exists an irreducible component Z of X_y such that it has dimension at least h. Now we only have to prove that Z contains at least one $x \in f^{-1}(y)$. However, we remember that $X_y \approx f^{-1}(y)$, hence necessarily Z contains an element x of $f^{-1}(y)$ and so Z is an irreducible component of $X_{f(x)} = X_y$ of dimension at least h.

Exercise 6. *Criterion of irreducibility.* This exercise uses results of the last exercise, see the hint for more details. Let k be a field.

(1) Suppose that $f \colon X \to Y$ is a map between finite type k-schemes, with Y being irreducible, such that every fiber is irreducible of a fixed dimension $d \geq 0$ (in particular f is surjective). Show that X is irreducible if
   (a) f is closed or,
   (b) X is equidimensional.
   Hint: write $X = \bigcup_i X_i$ *the decomposition into irreducible components of* X. *There is at least one irreducible component, say* X_1, *such that* $f(X_1)$ *is dense. Write* $U_1 = X_1 \setminus \bigcup_{i \neq 1} X_i$. *Using irreducibility of fibers, show that for every* $y \in f(U_1)$ *we have* $X_y = X_{1,y}$. *Use exercise 2 and hand-in item (3) to get an open set* $V \subset f(U_1)$ *of* Y *such that every fiber at* $y \in V$ *has dimension* $\dim(X) - \dim(Y)$. *Show also that* $X_i \setminus X_1 \subset f^{-1}(Y \setminus V)$. *Deduce that* X_1 *is the only*

irreducible component with a dense image. Conclude if you suppose (b). If you suppose (a), show that $f(X_1) = Y$ *and conclude.*

(2) Deduce that if X is an irreducible finite type k-scheme

$$X \times_k \mathbb{P}^n_k$$

is irreducible.

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AG II - Schemes and sheaves

Solutions– week 10

Exercise 1. *Functoriality of $\mathcal{O}(n)$.* Let R and S be $\mathbb{N}$-graded rings, where R is generated in degree 1, so that $\mathcal{O}(n)$ is a line bundle for each $n \in \mathbb{Z}$. Let $f\colon R \to S$ be an homogeneous map of degree $d \geq 1$, see Exercise 5, week 4. Denote by $g\colon U \to \operatorname{Proj}(R)$ the induced map at Proj from functoriality of Proj. Show then that for $n \geq 0$ we have $g^*\mathcal{O}(n) = \mathcal{O}(nd)_{|U}$.
Hint: check the claim on cocycles.
Solution key.
We first tacke the case where the the map is a graded ring map (homogeneous of degree 1).
As R is genererated in degree 1, $\mathcal{O}_{\operatorname{Proj}(R)}(1)$ is a line bundle. Therefore it's pullback to U is also a line bundle. Note that on U, because U is covered by D_+ of degree 1 elements that $\mathcal{O}_{\operatorname{Proj}(S)}(1)_{|U}$ is a line bundle. To check that they are isomorphic, we can compare cocycles. Denote by $(R_1)_h$ homogeneous elements of degree 1. Because

$$\bigcup_{r \in (R_1)_h} D_+(r) = \operatorname{Proj}(R)$$

then $U = \bigcup_{r \in (R_1)_h} D_+(r)$ by definition. Write $U_r = D_+(r)$. Then cocycles of both invertible sheaves are $\varphi_{rr'} = r/r'$.
Another way of seeing this is by what follows. Let $r \in (R_1)_h$. Define the multiplcation map

$$S_{(r)} \otimes R(n)_{(r)} \to S(n)_{(r)}$$

which glues to a map $g^*\mathcal{O}_{\operatorname{Proj}(R)}(1)_{|U} \to \mathcal{O}_{\operatorname{Proj}(S)}(1)_{|U}$ is seen to be a bijection sending an element s/r^d (with s homegeneous of degree $d+n$) to $s/r^{d+n} \otimes r^{d+n}/r^d$.
Now we tackle the general case of an homegeneous map of degree d. It suffices to address now the case of the isomorphism $\operatorname{Proj}(R) \to \operatorname{Proj}(R_d)$ by the inclusion of $v_d\colon R_d \to R$ the d-th Veronese subring. The claim follows from the fact that a degree n element in R_d is of degree nd in R. Indeed this leads to $(R_d(n))_{(r)} = R(nd)_{(r)}$.

Exercise 2. *A principal divisor is effective where it has no poles.* Let X be a Noetherian, normal and integral scheme. Recall that for normal Noetherian domain A, the ring A is the intersection of $A_{\mathfrak{p}}$ where $\operatorname{ht}(\mathfrak{p}) = 1$.
Let $f \in K(X)$. Let $U \subset X$ open. Show that if $\operatorname{div}(f)_{|U} \geq 0$ then $f \in \mathcal{O}_X(U)$. If $\operatorname{div}(f)_{|U} = 0$ then $f \in \mathcal{O}(U)^\times$.
Solution Key.
This is a local statement. So we can suppose that $U = \operatorname{Spec}(A)$ is affine. But then $A = \cap_{\mathfrak{p}} A_{\mathfrak{p}}$ where the intersection is taken on height 1 primes concludes.

Exercise 3. *Divisors that are not Cartier.* Let k be a field and $X = V(xy-zw)$ in $\mathbb{A}^4_k$. Note that X is integral and regular in codimension 1.

(1) Show that the closed subsets in X defined by $x = z = 0$ and $x = w = 0$ are prime divisors that are not Cartier. Denote by D_z and D_w these divisors.
(2) Show that $D_z + D_w$ is a Cartier divisor.

Solution key.
Note that D_z and D_w are prime Weil divisors isomorphic to $\mathbb{A}^2_k$ because

$$k[x, y, z, w]/(xy - zw, x, z) = k[y, w] \quad [x, y, z, w]/(xy - zw, x, w) = k[y, z].$$

If $I_z = (x, z)$ or $I_w = (x, w)$ where to define Cartier divisors, then these ideals would be locally principal, in the sense that in sufficiently small affine open sets, these ideals would be principal in the ring of functions of these opens. In particular, in each local ring these ideals would be principal. But at the local ring at the origin $\mathfrak{m}$, note that $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space of dimension 4, and the basis is given by the images of x, y, z, w. If I_z or I_w where to be generated by one element in this local ring, then the k-vector space spanned by the images of x, z and x, y respectively in $\mathfrak{m}/\mathfrak{m}^2$ would be of k-dimension 1, a contradiction.
On the other hand, note that $V(x) = D_z + D_w$, implying that this divisor is Cartier.

Exercise 4. *Exact sequence for class groups.* Let X be an integral separated scheme which is regular in codimension 1. Let Z be a proper closed subset of X and $U = X \setminus Z$.

(1) Show that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by $\sum n_i D_i \mapsto \sum n_i(D_i \cap U)$ is surjective.
(2) If $\mathrm{codim}(Z, X) \leq 2$, show that that this map is also injective.
(3) If $\mathrm{codim}(Z, X) = 1$ and Z is irreducible, show that there is an exact sequence
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 1$$
where $\mathbb{Z} \to \mathrm{Cl}(X)$ send 1 to Z.
(4) Let k be a field. Let Z be the zero set of an irreducible homogeneous polynomial of degree d in $\mathbb{P}^n_k$. Deduce that $\mathrm{Cl}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$.

Hint: You may look at chapter II.6 of Hartshorne.
Solution key.
The last statement follows frome example seeing that that $V_+(F)$ for F irreducible homegenous of degree d correspond to $\mathcal{O}(d)$ via the identification of Picard groups and Cartier class groups, and the $\mathcal{O}(d)$ is d times the generator of the Picard group which is infinite cyclic.

Exercise 5. Let A be a ring and $R_\bullet$ the graded ring $A[x_0, \dots, x_n]$ with $\deg(x_i) = 1$. Show that the natural map

$$R_m \to \Gamma(\mathrm{Proj}(R), \mathcal{O}(m))$$

is an isomorphism for $m \in \mathbb{Z}$. *Hint: Use the usual cover and the sheaf property.*

Solution key.
Consider the cover of Proj(A) by $D_+(x_i)$ for $i = 0, \dots, n$. Recall that

$$\mathcal{O}(m)_{|D_+(x_i)}(D_+(x_i)) = x_i^m R_{(x_i)}.$$

First note that the natural map is given by sending $f \in R_m$ to global section defined by $f \in x_i^m R_{(x_i)}$.
By the sheaf property, a global section of $\mathcal{O}(m)$ corresponds to a collection $f_i/x_i^{n_i}$ where $\deg(f_i) = m+n_i$ that agrees on intersections. Because we work in a polynomial ring, we can suppose that x_i does not divide f_i. Agreeing on intersections says that

$$f_i x_j^{n_j} = f_j x_i^{n_i}.$$

Because x_i, x_j is a regular sequence we deduce that $n_j = n_i = 0$. Therefore we deduce that $f_i = f_j = f$ an homogeneous element of degree m, concluding.

Exercise 6. *Support of coherent sheaves.* Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. We define

$$\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

1. Let A be a ring and M a finitely generated module. Show that $\operatorname{supp}(M)$ is closed.
2. In the same setup as in item (1), show that $\operatorname{supp}(M) = V(\operatorname{Ann}(M))$, where
$$\operatorname{Ann}(M) = \{f \in A \mid fM = 0\}.$$
3. Let A be a Noetherian ring, $f \in A$ and M be a finitely generated module. Show $\operatorname{Ann}(M)_f = \operatorname{Ann}(M_f)$.
4. Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Using the preceding point, define a quasi-coherent sheaf of ideals $\operatorname{Ann}(\mathcal{F})$. Show that $V(\operatorname{Ann}(\mathcal{F})) = \operatorname{supp}(\mathcal{F})$.

Solution key.

1. Let $m_1, \dots, m_r$ be generators of M. Note that the complement of $\operatorname{supp}(M)$ is the locus of $\mathfrak{p}$'s where $(m_i)_{\mathfrak{p}} = 0$ for all i. But if this holds this means that there is some $a \in A \setminus \mathfrak{p}$ such that $am_i = 0$. Then m_i is zero on $D_+(a)$, showing that the complement is open being the finite intersection of the loci where m_i's vanish.
2. We work on complements. We want to show that the complement of $\operatorname{supp}(M)$ is $\bigcup_{f \in \operatorname{Ann}(M)} D(f)$. Note that
$$\bigcup_{f \in \operatorname{Ann}(M)} D(f) \subset \operatorname{Spec}(A) \setminus \operatorname{supp}(M),$$
because if $M_f = 0$, then any further localization is zero. For the other inclusion, if $M_{\mathfrak{p}} = 0$, then if $f_i \in A \setminus \mathfrak{p}$ is an element killing m_i, then the product f of the f_i's is an element not in $\mathfrak{p}$ with $fM = 0$.
3. Note that $\operatorname{Ann}(M)_f \subset \operatorname{Ann}(M_f)$. If $(g/f^r)M_f = 0$, then $(g/f^r)m_i = 0$, implying that $gf^{n_i}m_i = 0$ for some n_i. Taking a big enough power shows the surjectivity of the map.
4. Immediate from last observation.

Remark. In this case, we then call $V(\mathrm{Ann}(\mathcal{F}))$ with it's natural scheme structure coming from the quasi-coherent sheaf of ideals $\mathrm{Ann}(\mathcal{F})$ the scheme theoretic support of $\mathcal{F}$.

Exercise 7. *Torsion free sheaves.* Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is torsion free if $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module for all opens $U \subset X$.

(1) Let $\mathcal{F}$ be any quasi-coherent sheaf. Show that $\mathcal{F}_{tors} \subset \mathcal{F}$, where $s \in \mathcal{F}_{tors}(U)$ if $s \mapsto 0$ along $\mathcal{F} \to \mathcal{F}_\eta$, is a quasi-coherent sheaf and that $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.
(2) Show that a map between torsion free sheaves is injective if and only if it is injective at a stalk at some point $x \in X$.
(3) Deduce that a map between locally free sheaves of rank 1 is injective or zero.

Solution key.

(1) It's defined to be the kernel of $\mathcal{F} \to \iota_\nu \mathcal{O}_{\mathrm{Spec}(K(X))}$ – the map ι_ν denotes $\mathrm{Spec}(K(X)) \to X$. It is therefore coherent as the kernel of a map between quasi-coherent sheaves.

    Note that being torsion free is a property that can be checked at stalks. This implies the second part of the statement.
(2) Follows from the fact that in the case the stalk map $\mathcal{F}(U) \to \mathcal{F}_\nu$ is injective.
(3) If the map is not injective, it will not be injective at the generic point but at the generic point we have a self map of $K(x)$-modules of $K(X)$ and the only way that this map is not injective is that this is the zero map. This concludes.

Exercise 8. *Generic flatness.* Let X be a reduced Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X.

(1) Show that there is an non empty open U such that $\mathcal{F}$ is locally free (possibly zero).

    Use Exercise 4.(3), week 9.
(2) Show by Noetherian induction[1] on X that there is a finite partition of X by locally closed subschemes (X_i) with the reduced scheme structure such that $\mathcal{F}$ is locally free when restricted (meaning taking the pullback) to X_i.

Solution Key.
The function $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$ takes a minimal value in $\mathbb{N}$ (could be zero). Because φ is semi-continuous, the locus where φ is equal to this minimal value is open. By the last part of the exercise on this topic, we get the claim. (Remark that there is no worry about connectedness.)
The previous claim shows the basis and the induction of Noetherian induction.

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[1] see *Hartshorne, II.3.16*

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AG II - Schemes and sheaves

Solutions – week 11

Exercise 1. *Tensor products,* Hom *and sheafification.*
Give examples of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that the tensor product presheaf and the presheaf

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$$

are not sheaves.
Hint: Play with $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ on projective spaces. Recall the computation of global sections of those, exercise 5, week 10.

Solution key. Take for example $\mathcal{O}(1) \otimes \mathcal{O}(-1)$ and $\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}(1))$. □

Exercise 2. *Effective Cartier divisors.* Let X be an integral scheme. A Cartier divisor on X represented by (f_i, U_i) is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i.

(1) Show, by looking at the ideal sheaf generated by the f_i's, that effective Cartier divisors are in one-to-one correspondence with ideal sheaves $\mathcal{I}$ that are a locally free sheaves of rank 1. We take this point of view in what follows.
(2) Let $\mathcal{L}$ be a locally free sheaf of rank 1. Show that $s \colon \mathcal{O}_X \to \mathcal{L}$ is non-zero if and only if the evaluation $\operatorname{ev}_s : \mathcal{L}^\vee \to \mathcal{O}_X$, defined by $\mathcal{L}^\vee(U) = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{L}_U, \mathcal{O}_u) \ni \varphi \mapsto \varphi(s)$ is injective.
(3) Fix a locally free sheaf $\mathcal{L}$ of rank 1. Deduce the following bijection,

$$\left(\Gamma(X, \mathcal{L}) \setminus \{0\}\right) / \mathcal{O}_X(X)^\times \to \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

that sends the class of a section s to $\operatorname{Im}(\operatorname{ev}_s)$.
(4) Suppose additionally that $\mathcal{O}_X(X)$ is a field. Show that if $\mathcal{L}$ is a locally free sheaf of rank 1 such that $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non zero section, then $\mathcal{L} \cong \mathcal{O}_X$.
Hint: in this case both $\mathcal{L}$ and $\mathcal{L}^\vee$ correspond to effective Cartier divisors.
(5) Additionally assume that X is normal, Noetherian and integral. *Two Weil divisors are called linearly equivalent if their difference is the divisor of some rational function.* Let D be a Weil divisor on X. Show that map sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $\operatorname{div}(f) + D$ gives a one to one correspondence

$$\frac{\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}}{\mathcal{O}_X(X)^\times} \to \{\text{Effective Weil divisors linearly equivalent to } D\}.$$

Careful, hypothesis does not imply that $\mathcal{O}_X(D)$ defined as

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid g \neq 0, (div(g) + D) \cap U \textit{ is effective}\}$$

is a line bundle. So you have to prove it independently of item (3).

Solution key. (1) If (f_i, U_i) is an effective Cartier divisor, then we see that defining $f_i\mathcal{O}_{U_i} \subset \mathcal{O}_{U_i}$ defines a sub-ideal sheaf of $\mathcal{O}_X$ by gluing because f_i/f_j are units in functions on the intersection by assumption so $f_i\mathcal{O}_{U_{ij}} = f_j\mathcal{O}_{U_{ij}}$. Reciprocally given a locally free ideal sheaf $\mathcal{I}$, we know that there is an open cover (U_i) with $\mathcal{I}_{U_i} = f_i\mathcal{O}_{U_i}$. Now, (f_i, U_i) defines an effective Cartier divisor. Say we choose other generators (on a possible different open cover, but we deal with this case by taking a common refinement) $\mathcal{I}_{U_i} = f_i'\mathcal{O}_{U_i}$. Then there is $g_i \in \mathcal{O}_{U_i}(U_i)^\times$ with $f_i = g_i f_i'$. This implies that $(f_i, U_i) = (f_i', U_i)$ has a Cartier divisor.

(2) This is a local check, so without loss of generality $\mathcal{L} = \mathcal{O}$ and $X = \operatorname{Spec}(A)$ is affine with A integral. So we are saying that $a \in A$ is non-zero if and only if

$$\begin{tikzcd} A \arrow[r, "1\mapsto \mathrm{id}"] \arrow[rr, bend right, "\cdot a"'] & \operatorname{Hom}_A(A, A) \arrow[r, "\mathrm{ev}_a"] & A \end{tikzcd}$$

the multiplication by a is injective.

(3) We produce an inverse. Say $\psi\colon \mathcal{I} \cong \mathcal{L}^\vee$. Then apply the $\operatorname{Hom}_{\mathcal{O}}(-, \mathcal{O})$ functor gives a map $s_\psi\colon \mathcal{O} \to \mathcal{I}^\vee \to \mathcal{L}$. Because an automorphism of a line bundle is always given by an element[1] in $\mathcal{O}_X(X)^\times$, it is clear that this inverse map does not depend on the choice of the isomorphism ψ.

(4) If both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero global sections, then say that $\mathcal{L} = \mathcal{I}$ an invertible ideal sheaf without loss of generality. But then $\mathcal{I}$ has a non-zero global section. Because we supposed that $\mathcal{O}_X(X)$ is a field, we see that 1 generated this ideal, concluding.

(5) About the inverse map, if D_1 is effective and linearly equivalent to D, this means that there exists $f \in K(X)$ such that $\operatorname{div}(f) + D = D_1$. So by definition f is a global section of $\mathcal{O}(D)$. Further details ommited. $\square$

Exercise 3. *Invertible sheaves and cocycles, a first encounter.* Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. Let (U_i) be a cover of X with trivializations $\varphi_i\colon \mathcal{L}_{U_i} \to \mathcal{O}_{U_i}$. We say that the associated cocycles $(\varphi \in \mathcal{O}_X(U_{ij})^\times)$ are defined to be $\varphi_i \circ \varphi_j^{-1}\colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ that we identify with $\varphi_{ij} \in \mathcal{O}_X(U_{ij})^\times$. Say $\mathcal{L}'$ is another invertible sheaf with associated cocycles (ψ_{ij}).

(1) Show that $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ and $\varphi_{ii} = 1$.

(2) Show that the cocycles (φ_{ij}^{-1}) associated with $\mathcal{L}^\vee$ are , and the cocycles associated with $\mathcal{L} \otimes \mathcal{L}'$ are $(\varphi_{ij}\psi_{ij})$.

(3) Show that if for every i there is some $h_i \in \mathcal{O}(U_i)^\times$ such that $h_i\varphi_{ij}h_j^{-1} = \psi_{ij}$, then $\mathcal{L} \cong \mathcal{L}'$.

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[1] An automorphism $\mathcal{L} \to \mathcal{L}$ is given locally by elements in $\mathcal{O}_{U_i}(U_i)^\times$ where $\mathcal{L}$ is trivial, but these will automatically glue. Indeed on the intersection they will be both equal to the restriction of the morphism.

We will go further in this study when introducing *first Čech cohomology.*

Solution key. (2) Denote suggestively $1/\varphi\colon \mathcal{O}_{U_i}$ the dual of φ^{-1}. This is a trivialization. Restricting to U_{ij} we see that the associated cocycle will be the dual of the multiplication by φ_{ij}^{-1} by functoriality of the dual. But the dual of the multiplication by an element identifies with the multiplication by this element. For the tensor product claim, note that the tensor product $\varphi_i \otimes \psi_j \colon \mathcal{L} \otimes \mathcal{L}' \to \mathcal{O}_X \otimes \mathcal{O}_X \xrightarrow{\text{mult}} \mathcal{O}_X$ is a trivialization.

(3) The condition ensures that the isomorphism $\psi_i^{-1} h_i \varphi_i \colon \mathcal{L}_{U_i} \to \mathcal{L}'_{U_i}$ glues.

□

Exercise 4. *Extension of coherent sheaves.* The goal is to show that if X is a Noetherian scheme, U an open subset and $\mathcal{F}$ is a coherent sheaf on U, then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}_{|U} \cong \mathcal{F}$.

(1) Show that on a Noetherian scheme X and $\mathcal{F}$ coherent sheaf, then if

$$\sum_i \mathcal{F}_i = \mathcal{F}$$

where $(\mathcal{F}_i)_{i\in I}$ are sub-coherent sheaves, then there exist a finite refinement $J \subset I$ such that $\sum_{j\in J} \mathcal{F}_j = \mathcal{F}$.

(2) Show that on a Noetherian affine scheme, every quasi-coherent sheaf is the direct colimit of it's coherent sub-sheaves. *Hint: Use the equivalence of categories with modules on global sections.*

(3) Let X be affine and $\iota\colon U \to X$ be an open subcheme. Show the claim in this case. *Hint: Show that $\iota_*\mathcal{F}$ is quasi-coherent, and then use a combination of (1) and (2) to conclude.*

(4) Show the claim in the general case of the statement of the exercise by induction on the number of open affines that are required to cover X. (Being covered by one open affine being the base case of the induction, and is the previous point. The rest is an induction play, see Hint.) *Hint: Say $X = X_1 \cup X_2$ where X_1 and X_2 are open subschemes that can be covered by strictly less open affines than X. By induction extend $\mathcal{F}_{X_1\cap U}$ to a coherent sheaf $\mathcal{G}_1$ defined on X_1. By gluing $\mathcal{F}$ and $\mathcal{G}_1$ it defines a coherent sheaf $\mathcal{G}'$ defined on $X_1 \cup U$. Now, extend $\mathcal{G}'_{|X_2\cap(X_1\cup U)}$ to a coherent sheaf $\mathcal{G}_2$ on X_2. Conclude by gluing $\mathcal{G}_1$ and $\mathcal{G}_2$ to a coherent sheaf on X.*

As an application, show that any quasi-coherent sheaf on a Noetherian scheme is a direct colimit of sub-coherent sheaves.

Solution key. (1) Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X. Suppose that

$$\sum_i \mathcal{F}_i = \mathcal{F}$$

where $(\mathcal{F}_i)_{i\in I}$ are sub-coherent sheaves. Then there exist a finite refinement $J \subset I$ such that $\sum_{j\in J} \mathcal{F}_j = \mathcal{F}$. Indeed, as we can cover X by *finitely* many open affines, we may prove that we can find a finite refinement for each open affine. But now this follows just from the fact that a coherent sheaf on a Noetherian affine scheme amounts to a finite module M. Each generator of M is a finite sum of sections of the $\mathcal{F}_i$'s.

(2) Recall that a Noetherian affine scheme $\operatorname{Spec}(A)$, quasi-coherent sheaves are equivalent to A-modules and coherent sheaves are equivalent to finite A-modules. Therefore to check that

$$\bigcup \mathcal{F}_\alpha = \mathcal{F}$$

it suffices to check it on global sections. But it amounts to say that an A-module is the union of it's sub finite A-modules.

(3) Cover U by *finitely* many open affine schemes $(U_i)_i$. Note that because an affine scheme is separated, intersections U_{ij} are also affine. We may use that affine morphism preserves quasi-coherence, and that quasi-coherent sheaves are stable by kernels. Denote by $\iota_i \colon U_i \to X$ and $\iota_{ij} \colon U_{ij} \to X$ inclusions. Now remark that

$$\iota_* \mathcal{F} = \ker \left(\bigoplus_i \iota_{i*} \mathcal{F} \to \bigoplus_{ij} \iota_{ij,*} \mathcal{F} \right)$$

where the map sends $(f_i) \mapsto (f_i - f_j)$.

We therefore know that this sheaf is the union of it's coherent subsheaves by (a)

$$\bigcup \mathcal{F}'_\alpha = \iota_* \mathcal{F}.$$

Because $\mathcal{F}_U$ is coherent, by the compacity remark, there exists finitely many $\alpha_1, \ldots, \alpha_n$ such that $\mathcal{F}_U$ is the sum of the $\mathcal{F}'_{\alpha_i,U}$. If we set $\mathcal{F}'$ to be the sum of these we get the claim.

(4) $\mathcal{G}$ is the union of it's coherent sub-modules, say $\mathcal{G}_\alpha$. By the compacity remark above, we may sum finitely many such that $\sum_i \mathcal{G}_{\alpha_i,U} = \mathcal{F}$.

(d) We proceed by induction on the number of affine schemes that can cover X. If X is affine, we are done. Otherwise, we can write

$$X = X_1 \cup X_2$$

with X_1 and X_2 open subschemes that can be written as union of stricly less open affines. Find a coherent sheaf on X_1 that we denote by $\mathcal{F}_1 \subset \mathcal{G}_{X_1}$ that extends $\mathcal{F}_{|X_1 \cap U}$ from $X_1 \cap U$ to X_1 by induction. Note that this defines a coherent sheaf on $X_1 \cup U$ by gluing $\mathcal{F}_1$ and $\mathcal{F}$. Denote this sheaf by $\mathcal{F}'_1$. Now extend $\mathcal{F}'_{1,|X_2 \cap (U \cup X_1)}$ from $X_2 \cap (U \cup X_1)$ to X_2 by induction to a coherent sheaf $\mathcal{F}_2$. Now remark that $\mathcal{F}_2$ and $\mathcal{F}'_1$ glue to the desired sheaf $\mathcal{F}'$.

(5) Let $s \in \mathcal{F}(U)$. Consider the coherent sheaf generated by s on U. Extend it to a coherent sheaf on X.

□

Exercise 5. *Divisors on regular curves.* Let k be an algebraically closed field. We say that C is a *regular k-curve over k* is a one dimensional separated, integral and regular scheme over k. Weil (=Cartier in this case) divisors are then of the form

$$D = \sum_i n_i x_i$$

for x_i being closed points of C. We define the *degree* of a divisor $D = \sum_i n_i x_i$ to be

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Let $f\colon C' \to C$ a finite k-morphism between regular k-curves. We define the *pullback* of an irreducible divisor (=closed point)

$$f^*x = \sum_{y \in C'_{cl} \text{ s.t. } x = f(y)} v_y(f^\sharp(t_x))y.$$

where $f^\sharp$ denotes the induced map at the local ring. Here, t_x denotes a generator of $\mathfrak{m}_x$ – this well defined because the choice of a generator is up to a unit. We extend f^* by linearity to $\mathrm{div}(C)$.

(1) Show that the pullback of a principal divisor is principal, implying that f^* factors through

$$f^*\colon \mathrm{Cl}(C) \to \mathrm{Cl}(C').$$

(2) Show that if the degree of the map $(= [K(C')\colon K(C)])$ is d, then $\deg(f^*D) = d\deg(D)$. *Hint: it suffices to show the claim for $D = x$ a closed point by linearity.*

(3) Assume now that C is also proper. Using the equivalence of categories seen in lecture on curves (that you can assume) between k-fields of k-transcendence degree 1 and regular proper k-curves, show that for every $t \in K(C)\backslash k$ we have a map $f_t\colon C \to \mathbb{P}^1_k$ from the inclusion $k(t) \subset K(C)$ such that $f^*(0 - \infty) = (f)$ where 0 denotes $V(f)$ in $\mathrm{Spec}(k[f]) \subset \mathbb{P}^1_k$ and ∞ denotes $V(1/f)$ in $\mathrm{Spec}(k[1/f]) \subset \mathbb{P}^1_k$. deduce that $\deg((f)) = 0$, and that therefore deg factor through

$$\deg\colon \mathrm{Cl}(C) \to \mathbb{Z}.$$

Solution key. (1) One sees that the pullback of $\mathrm{div}(g)$ for some $g \in K(C)$ is given by $\mathrm{div}(f^\sharp(g))$ where here $f^\sharp$ denotes the induced map at field of fractions.

(2) The fiber at x is a finite k-algebra of dimension d. Because k is algebraically closed, such algebras are isomorphic to

$$\prod k[t]/(t_i)^n_i$$

which has much factors has the set theoretic cardinality of the fiber and necessarily $\sum n_i = d$.

(3) If $t \in K(C) \setminus k$, because k is algebraically closed, t is transcendent. So $k(t) \to K(C)$ induces a map $C \to \mathbb{P}^1_k$ by the equivalence of categories. Now, it amounts to noticing that $\mathrm{div}(t) = 0 - \infty$ and using the preceding point.

□

Exercise 6. *Segre viewed with line bundles.* Fix an algebraically closed field k. Denote the projection of $\mathbb{P}^1 \times_k \mathbb{P}^1$ to the first and second factor by p_1 and p_2 respectively. View the first and second copy of $\mathbb{P}^1$ in the product as $\mathrm{Proj}(k[x_0, x_1])$ and $\mathrm{Proj}(k[y_0, y_1])$ respectively. Show that the global sections $p_1^*(x_i) \otimes p_2^*(y_j)$ for $0 \leq i, j \leq 1$ of $p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ give a closed embedding of $\mathbb{P}^1 \times_k \mathbb{P}^1$ in $\mathbb{P}^3$.

Solution key. Affine locally this is a Segre embedding. □

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Solutions – week 12

Exercise 1. *Homotopy invariance of class groups.* Let X be integral, Noetherian, separated and regular in codimension 1.

(1) Show that $X \times \mathbb{A}^1$ is also integral, Noetherian, separated and regular in codimension 1.
(2) Show that the projection π to the first component induces a morphism $\pi^* \colon \operatorname{Cl}(X) \to \operatorname{Cl}(X \times \mathbb{A}^1)$.
(3) Show that π^* is an isomorphism.

You may look at II.6.6 in Hartshorne.

Exercise 2. *A Künneth formula for class groups.* Let X be an integral, separated, Noetherian and locally factorial scheme. Let $n \geq 1$. Show that $\mathbb{P}^n_X = X \times \mathbb{P}^n_{\mathbb{Z}}$ also satisfies the above and that

$$\operatorname{Cl}(X \times \mathbb{P}^n_{\mathbb{Z}}) \cong \operatorname{Cl}(X) \times \mathbb{Z}.$$

Hint: Consider $\phi \colon \mathbb{P}^n_{K(X)} \to X \times \mathbb{P}^n_{\mathbb{Z}}$*. Show that* $\phi^* \colon \operatorname{Pic}(\mathbb{P}^n_X) \to \operatorname{Pic}(\mathbb{P}^n_{K(X)}) \cong \mathbb{Z}$ *gives a retraction of the first arrow in the exact sequence (exercise 8, week 9)*

$$\mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^n_X) \to \operatorname{Cl}(\mathbb{A}^n_X) \to 0$$

coming from the divisor $V_+(X_0)$ *in* $\operatorname{Cl}(\mathbb{P}^n_X)$*.*

Exercise 3. *Very ample divisors.* Let k be a field. Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 with $S_0 = k$. Denote by $X = \operatorname{Proj}(S)$. Suppose that X is integral and $\mathcal{O}_X(X) = k$.[1]

(1) Show that $\mathcal{O}_X(1)$ is k-very ample.
(2) If $\dim(X) \geq 1$, show that $\mathbb{Z} \xrightarrow{\mathcal{O}_X(1)} \operatorname{Pic}(X)$ is injective.
Hint: If $\mathcal{O}_X(1)$ *is torsion, it would imply that* $\mathcal{O}_X$ *is* k*-very ample.*
(3) If X is normal, deduce that if $0 \neq s \in \mathcal{O}_X(1)(X)$, then $\operatorname{div}(s) \in \operatorname{Cl}(X)$ has infinite order.

Solution key. (1) Denote by $s_0, \dots, s_n$ degree 1 elements that $S_0 = k$ generates S as an algebra. Note that $k[x_0, \dots, x_n] \to S$ sending $x_i \mapsto s_i$ is a graded surjection and therefore induces a closed immersion $\iota \colon X = \operatorname{Proj}(S) \to \mathbb{P}^n_k$. Note that by construction $\iota^* \mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_X(1)$. The claim now follows.

(2) If $\mathcal{O}_X(1)$ is torsion, meaning that $\mathcal{O}_X(n) \cong \mathcal{O}_X$, it would mean that $\mathcal{O}_X$ is k-very ample. As we suppose that $\mathcal{O}_X(X) = k$, this

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[1] This condition follows from previous assumptions if k is algebraically closed.

would imply that X is a point, a contradiction with the dimension hypothesis.

(3) Follows from the injection $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ (normal) and the previous point.

$\square$

Exercise 4. *Projective Cone.* This exercise is a follow-up to exercise 2, week 7.

Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 over S_0. Consider the $\mathbb{N}$-graded ring $S[t]$ with elements of S keeping their grading and with t placed in degree 1. We call $\mathrm{Proj}(S[t])$ with this grading the *projective cone.*

(1) Show that this grading comes from the product action

$$\mathbb{G}_{m,S_0} \times_{S_0} \mathrm{Spec}(S) \times_{S_0} \mathbb{A}^1_{S_0} \xrightarrow{(\mu_S \,\mathrm{pr}_{12}, \mu_{\mathbb{A}^1} \mathrm{pr}_{13})} \mathrm{Spec}(S) \times_{S_0} \mathbb{A}^1_{S_0}$$

where pr_{ij} denote projections, μ_S the action on $\mathrm{Spec}(S)$ and $\mu_{\mathbb{A}^1}$ the usual $\mathbb{G}_m$-action on $\mathbb{A}^1$.

(2) Show that there are natural identifications $V_+(t) = \mathrm{Proj}(S)$ and $D_+(t) = \mathrm{Spec}(S)$. Show furthermore that $V_+(S_+)$ (taken in $\mathrm{Proj}(S[t])$) identifies to the vertex (see exercise 6, week 10) in $\mathrm{Spec}(S)$. We therefore denote this closed subscheme by v.

(3) Let $s_0, \dots, s_n$ be generators of S in degree 1. Show that $\mathrm{Proj}(S[t]) \setminus v$ is covered by the open sets $D_+(s_i)$ and that each open set is isomorphic to $\mathrm{Spec}(S_{(s_i)}[t])$. Deduce that we have a natural map

$$p\colon \mathrm{Proj}(S[t]) \setminus v \to \mathrm{Proj}(S).$$

(4) Let k be an algebraically closed field, and suppose $S_0 = k$. Suppose that S is integral, Noetherian and normal. Suppose that $X = \mathrm{Proj}(S)$ is of dimension ≥ 1. Show that p^* induces an isomorphism on class groups. Deduce that, if $C = \mathrm{Spec}(S)$ denotes the cone of X then we have an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(C) \to 1$$

where the first morphism sends 1 to the class of $\mathcal{O}_X(1)$, and the second is the composition of p^* and the restriction to C.

Proof. (1) Analyze that degree 1 elements are still of degree 1.

(2) The $V_+(t)$ asserion is immediate. For the $D_+(t)$ it amounts to realizing that degree zero elements of S_t are just S. For the last assertion, note that $\mathrm{Proj}(S_0[t])$ identifies with $\mathrm{Spec}(S_0)$.

(3) Note that $(s_0, \dots, s_n)$ generates S_+ as an ideal. Therefore the claim on the cover follows. Note that the degree zero part of $S[t]_{s_i}$ identifies to $S_{(s_i)}[\frac{t}{s_i}]$ which gives the claim.

(4) We denote by $\overline{X}$ the projective cone.

We show that p^* induces an isomorphism on class groups. The previous point shows that $K(\overline{X}) = K(X)(T)$ for $T = t/s_i$ for some i. From this observation, one can actually apply the same proof as proposition chapter 2, 6.6 in Hartshorne. We give a bit more details. We use the terminology this proof in what follows. We can separate the codimension points of $\overline{X} \setminus v$ in two distinct families. Type 1 points will be points such that the image in X is of codimension 1. Type 2 are points such that the image is generic. These points are in one to one correspondence with codimension points of $\mathbb{A}^1_{K(X)}$ along the dominant map $\mathbb{A}^1_{K(X)} \to \overline{X} \setminus v$. We can define π^* exactly as in proposition 6.6 at the level of divisors, and it gives a surjection to the subgroup of $\mathrm{Cl}(\overline{X} \setminus v)$ generated by type 1 points. Note that the exact same proof as the one in proposition 6.6 shows that any type 2 points are linearly equivalent to type 1 points. This shows surjectivity. Injectivity also is proven with the exact same argument.

Note also that by exercise 2.3 and the exercise 4, week 10 we have an exact sequence (where $1 \in \mathbb{Z}$ is sent to $\mathcal{O}_X(1)$ seen as $V_+(t)$, which is prime because X is integral)

$$1 \to \mathbb{Z} \to \mathrm{Cl}(\overline{X}) \to \mathrm{Cl}(C) \to 1$$

because as argued above $D_+(t) = \mathrm{Spec}(S) = C$.

Note that because $v \in \overline{X}$ is at least of codimension 2, we have $\mathrm{Cl}(\overline{X}) \cong \mathrm{Cl}(\overline{X} \setminus v)$. We can now use that p^* induces an isomorphism to conclude.

□

Exercise 5. *Computations of class groups on quadric hypersurfaces.* Suppose that k is algebraically closed and $char(k) \neq 2$. Let $2 \leq r \leq n$. Consider the ring (equipped with the standard grading)

$$S_r = k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2).$$

You can assume that this ring is normal. (See Hartshorne, exercise 6.4 for a proof).

(1) Show that up to a linear change of variable, we can suppose that

$$S_r = k[x_0, \dots, x_n]/(x_0 x_1 + x_2^2 \cdots + x_r^2).$$

Denote by $C_r = \mathrm{Spec}(S_r)$ and $X_r = \mathrm{Proj}(S_r)$.

(2) Show that $\mathrm{Cl}(C_r)$ is cyclic when $r \neq 3$
Hint: Consider the prime divisor $V(\sqrt{(x_1)})$ and the exact sequence of week 9, exercise 8.

(3) Show that $\mathrm{Cl}(C_2) \cong \mathbb{Z}/2\mathbb{Z}$.
Hint: Consider the same exact sequence. See Hartshorne example 6.5.2.

(4) Show that $\mathrm{Cl}(C_3) \cong \mathbb{Z}$.
Hint: show that after a suitable change of variable we see that $X_r \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. Then use exercise 1 and the exact sequence of the last point of the above exercise.

(5) Show that $\mathrm{Cl}(C_r) \cong 0$ of $r \geq 4$. In particular, S_r is factorial.
Hint: show that (x_1) is prime in this case and conclude.

(6) Use the exact sequence of the last point of the above exercise to compute $\mathrm{Cl}(X_r)$ for all $r \geq 2$.

Solution key.

1. Note that $x_0^2 + x_1^2 = (x_0 + ix_1)(x_0 - ix_1)$ where $i \in k$ is a root of -1 in k. Because $char(k) \neq 2$ we can set $y_0 = x_0 + ix_1$ and $y_1 = x_0 - ix_1$ two different variables.
2. If $r \neq 3$ note that $V(x_1)$ is irreducible. We can then use the same strategy as in example 6.5.2 in Hartshorne.
3. See example 6.5.2 in Hartshorne.
4. Up to a change of variable we recognize the Segre embedding of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ in $\mathbb{P}^3_k$. We may work with
$$S_r = k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3)$$
Note that in the exact sequence from the previous exercise, $1 \in \mathbb{Z}$ is sent to the Cartier corresponding to $p_1^*\mathcal{O}_(1) \otimes p_2^*\mathcal{O}(1)$ when we view this in the product. In other words, this is the class of $(1, 1) \in \mathrm{Cl}(\mathbb{P}^1_k \times_k \mathbb{P}^1_k) = \mathbb{Z} \oplus \mathbb{Z}$. The result follows.
5. $V(x_1)$ is principal and prime.
6. Follows from the exact sequence.

$\square$

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Solutions – week 13

Exercise 1. *Separable extensions and differentials.* Let k be a field and l a finite extension. Show that $\Omega^1_{l|k} = 0$ if and only if l is a separable extension.

Solution key. If l is separable and finite then $l = k(\alpha)$ for some algebraic and separable element α. But then $l = k[t]/(f(t))$ and $f'(t)$ is not zero in the quotient by separability. This concludes one way by the conormal sequence. For the other direction, if l is not separable, then there is some $\alpha \in l$ such that $l'(\alpha) = l$ for some sub-extension l' such that the extension is not separable implying that $l = l'[t]/(f(t))$ with the derivative vanishing in this quotient. Therefore $\Omega_{l|l'} = l$ by the conormal sequence. But if $\Omega_{l|k} = 0$ then $\Omega_{l|l'} = 0$ by the fundamental sequence of cotangent sheaves, which is a contradiction.

□

Exercise 2. *Derivations on an elliptic curve.* Let R be a ring, $P = R[x_1, \dots, x_n]$ and $P \to A$ a surjection, with kernel I. Recall that by the conormal sequence, if $d\colon I/I^2 \to \bigoplus_{i=1}^n A dx_i$ is given by sending a polynomial to the image of it's derivative then we have an exact sequence

$$I/I^2 \to \bigoplus_{i=1}^{n} A dx_i \to \Omega^1_{A|R} \to 0.$$

We denote by $\mathrm{T}^1_{A|R} = \mathrm{Hom}_A(\Omega^1_{A|R}, A) = \mathrm{Der}_R(A, A)$, the *$A$-module of R-derivations of A.*

(1) Let
$$E = \mathrm{Proj}(\mathbb{C}[X, Y, Z]/(Y^2 Z - (X^3 + Z^3))).$$
Denote by x, y the images of $\frac{X}{Z}, \frac{Y}{Z}$ in $A_Z := \mathcal{O}_E(D_+(Z))$ and s, t the images of $\frac{X}{Y}, \frac{Z}{Y}$ in $A_Y := \mathcal{O}_E(D_+(Y))$. Show using the sequence recalled above that (meaning that any derivation is a scalar multiplication of the written generator)
$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}).$$

(2) Moreover show that the generators displayed above agree on the intersection $D_+(YZ)$, giving a non-vanishing global section π of $\mathrm{T}_{E|\mathbb{C}} := \mathrm{Hom}_{\mathcal{O}_E}(\Omega^1_{E|\mathbb{C}}, \mathcal{O}_E)$ implying that
$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

Solution key. (1) We write functions on $D_+(Z)$ and $D_+(Y)$. They are

$$\mathbb{C}[x', y']/(y'^2 - (x'^3 + 1)) \quad \mathbb{C}[s', t']/(t' - t'^3 - s'^3)$$

where x', y', s', t' denotes $\frac{X}{Z}, \frac{Y}{Z}$ and $\frac{X}{Y}, \frac{Z}{Y}$ before taking the quotient. By the conormal sequence we have

$$\Omega_{A_Z|k} = \frac{A_Z dx' \oplus A_Z dy'}{-3x^2 dx' + 2y dy'} \quad \Omega_{A_Y|k} = \frac{A_Y ds' \oplus A_Y dt'}{-3s^2 ds' + (1-3t^2) dt'}$$

We are interested in the dual of both these modules. We see the dual as a submodule of the dual of $A_Z dx' \oplus A_Z dy'$ and $A_Y ds' \oplus A_Y dt'$ respectively. Using the identification with derivations, we write the dual basis (dx', dy') and (ds', dt') as $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$. The claimed generators are indeed in this submodule. We want to show that they generate. So let $f_1, f_2 \in A_Y$ such that

$$f_1(-3x^2) + f_2 2y = 0.$$

So $f_1 3x^2 = f_2 2y$. But $A_Z/y = \mathbb{C}[x']/(x'^3 + 1)$ and $3x^2$ is invertible in this ring (it's not a root of the polynonial). So we see that $2y \mid f_1$. Also $A_Y/x^2 = \mathbb{C}[x', y']/(x^2, y^2 - 1)$ and a similar argument holds to conclude that $3x^2 \mid f_2$. So we have $f_1 = 2y\lambda_1$ and $f_2 = 3x^2\lambda_2$. Therefore $2y\lambda_1 3x^2 = 3x^2\lambda_2 2y$. We can simplify to get $\lambda_1 = \lambda_2$, which concludes. The reasoning for the second module is similar.

Also, note that because rings that we are dealing with are integral, we necessarily have that the map $A_Z \to A_Z\frac{\partial}{\partial x} \oplus A_Z\frac{\partial}{\partial y}$ sending 1 to the generator is injective. Same holds for the second module. We have therefore concluded that

$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t})$$

are free sheaves of rank 1.

(2) It suffices to show that both derivations agree on the intersection. Note that $x = st^{-1}$ and $y = t^{-1}$. Now,

$$\left((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(st^{-1}) = \frac{3t^2 - 1}{t} + \frac{3s^3}{t^2}.$$

But this equals, because $s^3 = t - t^3$, to $2/t = 2y$. Also

$$\left((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}\right)(t^{-1}) = \frac{3s^2}{t^2} = 3x^2,$$

which concludes that derivations indeed agree on the intersection.

Last statement follows because $T^1_{E|k}$ is an invertible sheaf by the first point and that we found a global nowhere vanishing global section.

□

Exercise 3. *Relative* Spec. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. This means that it is a sheaf $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{A}(V)) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{A}(U)) \arrow[d] \\
V \arrow[r] & U
\end{tikzcd}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\mathrm{Spec}(\mathcal{A}(U_i)))$ to an S-scheme

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \to S.$$

(3) Show that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ satisfies the following universal property in the category of S-schemes. If $f\colon T \to S$ is an S-scheme then a S-morphism $T \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ is the same as a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. Deduce that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is independent of the affine cover for the construction.

(4) Let $f\colon X \to Y$ be an affine morphism of schemes. Show that there is a natural isomorphism of Y-schemes $X \cong \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$.

(5) Let $\mathcal{E}$ be a locally free sheaf of finite rank on S. We define

$$\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E}^\vee))$$

where the $\mathcal{O}_S$-algebra $\mathrm{Sym}(\mathcal{E}^\vee)$ denotes the free $\mathcal{O}$-algebra on $\mathcal{E}^\vee$.[1] Show that a S-morphism from $f\colon T \to S$ to $\mathbb{V}(\mathcal{E})$ is the same as a global section of $f^*(\mathcal{E})$, *i.e* an element of $f^*(\mathcal{E})(T)$.

(6) Show that there is always a canonical section of $p\colon \mathbb{V}(\mathcal{E}) \to S$ which correspond to $0 \in \mathcal{E}(S)$ which defines a closed subscheme of $\mathbb{V}(\mathcal{E})$ isomorphic to S. We call this closed subscheme the *zero section of* $\mathbb{V}(\mathcal{E})$.

Solution key. (1) We may cover V by principal opens affine to prove the isomorphism locally. In this case it is clear that it follows from quasi-coherence.

(2) Using the above we see that $\mathrm{Spec}(\mathcal{A}(U_{ij})) \to \mathrm{Spec}(\mathcal{A}(U_i))$ are open immersions, being locally pullbacks of open immersions. It follows that we can glue this to a scheme.

(3) Cover S by affine schemes U_i. It induces an open cover of T by open subschemes T_i. Suppose we are given $f\colon T \to \underline{\mathrm{Spec}}_S(\mathcal{A})$. Note that this corresponds by gluing to a collection of maps of U_i-schemes $f\colon T_i \to \mathrm{Spec}(\mathcal{A}(U_i))$ that appropriately glues. This correspond one to one to a collection of $\mathcal{O}(U_i)$-algebra maps $\mathcal{A}(U_i) \to \mathcal{O}(T_i)$ which correspond one to one to morphisms of $\mathcal{O}_S$-algebras $\mathcal{A} \to f_*\mathcal{O}_T$, from which the claim follows by adjunction.

(4) As f is affine, note that $f_*\mathcal{O}_X$ is quasi-coherent. Note also that for an open affine $U \subset Y$, we have a natural identification $f^{-1}(U) = \mathrm{Spec}(\mathcal{O}(f^{-1}(U))) = \mathrm{Spec}(f_*\mathcal{O}_X(U))$ from which the claim follows.

(5) By the above, such a morphism is the same as the data of an $\mathcal{O}_T$-algebra morphism $\mathrm{Sym}(f^*\mathcal{E}^\vee) \to \mathcal{O}_T$. Therefore this the same as an

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[1] It's a gluing of the usual construction in liner algebra.

$\mathcal{O}_T$-module morphism $f^*\mathcal{E}^\vee \to \mathcal{O}_T$. By duality, this is the same as a section of $f^*\mathcal{E}$.

(6) Affine locally on S, say on some affine $\operatorname{Spec}(R)$, we can assume that $\mathcal{E} \cong \widetilde{R^n}$ is finite free, and the zero section correspond to the origin of $\mathbb{A}^n_R$.

□

Remark. If $S = \operatorname{Spec}(k)$ and V a finite-dimensional vector space, then $\mathbb{V}(V)$ is the scheme-theoretic incarnation of the k-vector space V.

Exercise 4. *Projective bundles.* Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $\mathcal{E}$ be a locally free sheaf of finite rank on S.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{tikzcd}
\operatorname{Proj}(\mathcal{A})(V)) \arrow[r] \arrow[d] & \operatorname{Proj}(\mathcal{A})(U)) \arrow[d] \\
V \arrow[r] & U
\end{tikzcd}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\operatorname{Proj}(\mathcal{A})(U_i)))$ to an S-scheme (*the relative* Proj)

$$\pi \colon \underline{\operatorname{Proj}}(\mathcal{A}) \to S.$$

When $\mathcal{A} = \operatorname{Sym}(\mathcal{E}^\vee)$ we denote $\underline{\operatorname{Proj}}(\operatorname{Sym}(\mathcal{E}^\vee)) = \mathbb{P}(\mathcal{E})$, the *projective bundle associated to* $\mathcal{E}$.

(3) Show that $\mathbb{P}(\mathcal{E})$ satisfies the following universal property in the category of S-schemes. If $f \colon T \to S$ is an S-scheme then a S-morphism $T \to \mathbb{P}(\mathcal{E})$ is the same as a sub-line bundle[2] $\mathcal{L} \subset f^*\mathcal{E}$.
Hint: Show that the line bundles $\mathcal{O}(1)$ *on* $\operatorname{Proj}(\operatorname{Sym}(\mathcal{E}^\vee)(U))$ *glue naturally to a line bundle* $\mathcal{O}(1)$ *with a surjection*

$$\pi^*\mathcal{E}^\vee \to \mathcal{O}(1).$$

The identity correspond therefore to the dual inclusion $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$. *Recall that for locally free sheaves of finite rank, pullback and dual naturally commute.*

(4) Show that the surjection

$$\operatorname{Sym}(\mathcal{E}^\vee \oplus \mathcal{O}_S) \to \operatorname{Sym}(\mathcal{E}^\vee)$$

induces a closed immersion

$$\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$$

and that the open complement identifies to $\mathbb{V}(\mathcal{E})$, leading to an open-closed decomposition

$$\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) = \mathbb{V}(\mathcal{E}) \sqcup \mathbb{P}(\mathcal{E}).$$

Remark. This generalizes the open closed decomposition $\mathbb{P}^{n+1}_k = \mathbb{A}^{n+1}_k \sqcup \mathbb{P}^n_k$. We can therefore interpret $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ as a compactification

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[2] a subsheaf which is a line bundle, and such that $f^*\mathcal{E}/\mathcal{L}$ is locally free.

of $\mathbb{V}(\mathcal{E})$ where we add an ∞-point to each line in $\mathbb{V}(\mathcal{E})$, namely the corresponding point in $\mathbb{P}(\mathcal{E})$.

(5) Show that $\mathcal{O} \subset \mathcal{E} \oplus \mathcal{O}$ defines a section of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \to S$ which leads to an open-closed decomposition

$$\mathbb{V}(\mathcal{O}(1)) \sqcup S = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}).$$

Remark. This generalizes

$$\mathbb{P}_k^{n+1} = \left(\bigcup_{i=0}^{n} D_+(x_i) \right) \sqcup [0 : \ldots 0 : 1].$$

Solution key. (1) Analogous to the above.

(2) Same.

(3) We give a proof that does not use the proposition about morphism to $\mathbb{P}_A^n$ case. In fact, it is a reformulation of this proof in other terms and a more general setup. We try take profit as much as we can of the known properties of the Proj construction, in particular it's functoriality studied in a previous exercise.

(a) *Some facts on* $\underline{\mathrm{Proj}}_S$. Let $\mathcal{F}$ be a finite locally free sheaf on S. We denote by $\pi \colon \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F})) \to S$ the structure map.

Claim. We have a canonical map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ which is an isomorphism. By adjunction, we get a canonical map $\pi^*\mathcal{F} \to \mathcal{O}(1)$. This last map is surjective.

Proof. Note first that for $\mathcal{F} = \mathcal{O}_S^{\oplus n}$, the claim follows from the calculation of the global sections of $\mathcal{O}(1)$ on the projective space, namely it is it's natural reinterpretation. We are going construct a map $\mathcal{F} \to \pi_*\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F}))}(1)$ which is functorial in $\mathcal{F}$ and this will allow to conclude as in exercise 2, week 9, the exercise on duals.

We may define the natural map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ affine locally on S, because it will readily glue. So say S is affine, where $\mathcal{F}$ is finite free. We may cover $\mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$ by $D_+(f)$ for $f \in \mathcal{F}(S)$. We have

$$\mathcal{O}(1)(D_+(f)) = \mathrm{Sym}(\mathcal{F})(1)_{(f)}.$$

Therefore we can define the natural map $\mathcal{F} \to \pi_*\mathcal{O}(1)$ by sending $g \in \mathcal{F}(S)$ to the unique global section of $\mathcal{O}(1)$ that restricts locally to $(g \in \mathcal{O}(1)(D_+(f)))_{f \in \mathcal{F}}$. Because this map is natural[3] we can conclude using the local free case that we already know as explained above.

Affine locally on S, and on an open $D_+(f)$ the surjection $\pi^*\mathcal{F} \to \mathcal{O}(1)$ reads as

$$\mathrm{Sym}(\mathcal{F})_{(f)} \otimes \mathcal{F} \to \mathrm{Sym}(\mathcal{F})(1)_{(f)} \quad 1 \otimes f \to f$$

---

[3] If some reader want some language, we are defining a natural transformation between the identity functor on locally free sheaves and the functor sending $\mathcal{F}$ to $\pi_*\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{F}))}(1)$.

which is surjective.
Taking duals of the previous point, we get,
Corollary. *On $\mathbb{P}(\mathcal{E})$, we have a natural inclusion $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \subset \pi^*\mathcal{E}$.*
See the next exercise for more on this inclusion.

$\square$

(b) Let $\mathcal{L}$ be a line bundle on S.
Claim. *The map $\pi\colon \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L})) \to S$ is an equality. Moreover if $s \in \mathcal{L}(S)$ is a global section, then $D_+(s)$ in $\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L}))$ correspond to $D(s)$ in S. Moreover $\mathcal{L}$ corresponds to $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L}))}(1)$ by point (a).*
We may work locally on S where $\mathcal{L} = \mathcal{O}_S t$ for a generator $t \in \mathcal{L}(S)$. So $\mathrm{Sym}(\mathcal{L}) = \mathcal{O}_S[t]$. But then $\underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{L})) = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[t]_{(t)}) = \underline{\mathrm{Spec}}_S(\mathcal{O}_S)$. Indeed, the natural inclusion $\mathcal{O}_S \subset \mathcal{O}[t]_{(t)}$ is an equality. The claim about $D_+(s)$ and $D(s)$ also follows from the previous inspection.

(c) We now proceed to the proof of the statement of the exercise. Let $T \xrightarrow{f} S$ be an S-scheme. We define a functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$

$$P(T \xrightarrow{f} S) = \{\mathcal{L} \subset f^*\mathcal{E} \ \mid \mathcal{L} \text{ is a line bundle, and } f^*\mathcal{E}/\mathcal{L} \text{ is locally free}\}$$

The functoriality is defined as follows. Let $f\colon T \to S$ and $f'\colon T' \to S$ be S-schemes. If $g\colon T' \to T$ is a morphism of S-schemes we define $P(g)$ to be $\mathrm{Im}(g^*\mathcal{L} \to f'^*\mathcal{E})$ that we may abbreviate as $g^*\mathcal{L} \subset f'^*\mathcal{E}$.
We may write $P(T)$ and let $f\colon T \to S$ implicit. We want to show that there is a natural bijection

$$P(T) \cong \mathrm{Sch}_S(T, \mathbb{P}(\mathcal{E})).$$

As $T \times_S \mathbb{P}(\mathcal{E}) = \mathbb{P}(f^*\mathcal{E})$, we have $\mathrm{Sch}_S(T, \mathbb{P}(\mathcal{E})) \cong \mathrm{Sch}_T(T, \mathbb{P}(f^*\mathcal{E}))$ by sending a map $T \to \mathbb{P}(f^*\mathcal{E})$ to $T \to \mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ so we can suppose that $T = S$ and $f = \mathrm{id}$. We now define a map

$$\alpha\colon P(S) \to \mathrm{Sch}_S(S, \mathbb{P}(\mathcal{E}))$$

by sending $\mathcal{L} \mapsto (S = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{E}))$. The first arrow is the equality explained in point (b) and the second arrow comes from the surjection $\mathcal{E}^\vee \to \mathcal{L}^\vee$ (dual to the given inclusion $\mathcal{L} \subset \mathcal{E}$) and the functoriality of $\underline{\mathrm{Proj}}_S$. We want to show that this is a bijection. To this end we define an inverse map β. Given a section $g\colon S \to \mathbb{P}(\mathcal{E})$, we get using the natural inclusion $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$ an inclusion $g^*\mathcal{O}(-1) \subset \mathcal{E}$. This is our β.

- Note that $\beta \circ \alpha = \mathrm{id}$ because the natural surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$ pullback *via* $(S = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{E}))$ to $\mathcal{E}^\vee \to \mathcal{L}^\vee$ by functoriality of Proj and (a) and (b) above.
- We now show that $\alpha \circ \beta = \mathrm{id}$, this will conclude the proof. Let $g\colon S \to \mathbb{P}(\mathcal{E})$ a section, meaning an S-scheme map, meaning $\pi \circ g = \mathrm{id}$. We want to show that the following

diagram commutes

$$\begin{tikzcd} & \mathbb{P}(g^*\mathcal{O}(-1)) \arrow[d] \\ S \arrow[ur, equal] \arrow[r, "g"] & \mathbb{P}(\mathcal{E}) \end{tikzcd}$$

which basically means that we can identify g to the map obtained by functoriality of Proj. This is actually a direct consequence of the definition of pullback os $\mathcal{O}_S$-modules and the functoriality of Proj, but we try to write it down carefully in what follows.
To see that it is the case, we may work affine locally on S; let's write then $\mathcal{F}(S) = M$. The map induced by Proj comes from the pullback by g of the natural map $\pi^* M^\vee \to \mathcal{O}(1)$, which is a map $M^\vee \to g^*\mathcal{O}(1)$. We denote the image by this map of some $\phi \in M^\vee$ by $g^*\phi$. The map g is given locally by compatible ring maps

$$g^\sharp \colon \operatorname{Sym}(M^\vee)_{(\phi)} \to \mathcal{O}_S((D(g^*\phi))).$$

Note that on $D(g^*\phi)$ the line bundle $g^*\mathcal{O}(1)$ is trivial. Indeed on $D_+(\phi)$ it is equal to $\operatorname{Sym}(M^\vee)_{(\phi)}\phi$, so the pullback is equal to $\mathcal{O}_S(D(g^*\phi))g^*\phi$. Also note that $g^\sharp(\frac{\psi}{\phi})g^*\phi = g^*\psi$ in $g^*\mathcal{O}(1)(D(g^*\phi))$ because $\frac{\psi}{\phi}\phi = \psi$ in $\operatorname{Sym}(M^\vee)_{(\phi)}\phi = \mathcal{O}(1)(D_+(\phi))$. But now, it immediately follows that the map induced by Proj is locally given by

$$\operatorname{Sym}(M^\vee)_{(\phi)} \to \operatorname{Sym}(g^*\mathcal{O}(1))_{(g^*\phi)} = \mathcal{O}_S(D(g^*\phi))[g^*\phi]_{(g^*\phi)} = \mathcal{O}_S(D(g^*\phi))$$

which sends $\frac{\psi}{\phi}$ to $\frac{g^*\psi}{g^*\phi} = \frac{g^\sharp(\frac{\psi}{\phi})g^*\phi}{g^*\phi} = g^\sharp(\frac{\psi}{\phi})$, which concludes.
Remark. If $S = \operatorname{Spec}(k)$ and V a finite-dimensional vector space, then $\mathbb{P}(V)$ is the scheme-theoretic incarnation of the projective space of V.
Remark. If $\mathcal{E} = \mathcal{O}_S^{\oplus n+1}$, then we get a generalization of statement seen in class over an affine base of the universal property of $\mathbb{P}^n_S$. Let $T \to \mathbb{P}^n_S$ be a map. Because we have chosen a basis, we have a canonical identification between sub-line bundles $\mathcal{L}' \subset \mathcal{O}_T^{\oplus n+1}$ and quotients $\mathcal{O}_T^{\oplus n+1} \to \mathcal{L}$ – this second interpretation is therefore the same as the choice of $n+1$ globally generating sections of $\mathcal{L}(T)$ (up to $\mathcal{O}_S^\times$). We explain why it is a very good idea to then denote the induced morphism by

$$f \colon T \xrightarrow{[s_0 : \cdots : s_n]} \mathbb{P}^n_S.$$

Let $x \in T$ be a point. Then $f(x) \in \mathbb{P}^n_{k(x)}$ which is the set of lines in $k(x)^{\oplus n+1}$. We claim that

$$[s_0(x) \colon \ldots : s_n(x)]$$

defines a line in $k(x)^{\oplus n+1}$. Note that by definition $s_i(x) \in \mathcal{L}(x)$ which is not canonically identified with $k(x)$, so we need to explain how to interpret $s_i(x)$ as an element of $k(x)$. Actually we will not claim that this is a well defined element, however we claim that $[s_0(x)\colon \dots : s_n(x)]$ is a well defined line. Because the sections globally generate, $x \in D(s_i)$ for some i say $i = 0$. So we may triviliaze on this open $\frac{1}{s_0}\colon \mathcal{L}_{D(s_0)} \to \mathcal{O}_{D(s_0)}$. Using this trivialization we interpret

$$[s_0(x)\colon \dots : s_n(x)] \quad \text{as} \quad [1 : \frac{s_1}{s_0}(x)\colon \dots : \frac{s_n}{s_0}(x)].$$

This indeed defines a line in $k(x)^{\oplus n+1}$ and one can check that it does not depend on the trivialization. Also note that to understand the image of the dual

$$k(s)^{\oplus n+1} \xrightarrow{(s_0(x)\colon \dots\colon s_n(x))} \mathcal{L}(x),$$

one can use the above trivialization around x, and then dualize, and the one finds that the line $f(x)$ in question is indeed the one $[s_0(x)\colon \dots s_n(x)]$ whose construction is explained above.

(4) The open complement is given by $D_+(t)$, if t designates the global section $(0, 1) \in \mathcal{E} \oplus \mathcal{O}_S$. More precisely, we identify $\mathrm{Sym}(\mathcal{E}^\vee \oplus \mathcal{O}_S) = \mathrm{Sym}(\mathcal{E}^\vee)[t]$. Therefore the degree zero part of the localization by t identifies with $\mathrm{Sym}(\mathcal{E}^\vee)$.

(5) We precise that $\mathbb{V}(\mathcal{O}(1))$ designates the bundle on $\mathbb{P}(\mathcal{E})$.

Note first that the closed subscheme corresponding to $\mathcal{O}_S \subset \mathcal{E} \oplus \mathcal{O}_S$ is given by the graded ideal sheaf $\langle \mathcal{E}^\vee \rangle \subset \mathrm{Sym}(\mathcal{E}^\vee)[t]$. Indeed the above inclusion correspond by duality to the surjection $\mathcal{E}^\vee \oplus \mathcal{O}_S \to \mathcal{O}_S$ and the closed subscheme corresponding is given by the graded ideal sheaf generated by the kernel of this map.

Note that the the graded inclusion $\mathrm{Sym}(\mathcal{E}^\vee) \to \mathrm{Sym}(\mathcal{E}^\vee)[t]$ induces by functoriality of Proj a morphism $U \to \mathbb{P}(\mathcal{E})$ where U is the open subscheme complement of the closed subscheme mentioned above.

Our goal is now to show that $U \to \mathbb{P}(\mathcal{E})$ is isomorphic as $\mathbb{P}(\mathcal{E})$-scheme to $\mathbb{V}(\mathcal{O}(1)) \to \mathbb{P}(\mathcal{E})$. To this end we view these as functors on $\mathbb{P}(\mathcal{E})$-schemes *via* the Yoneda embedding. It suffices therefore to construct a natural isomorphism of their respective functor of points. Recall that a $\mathbb{P}(\mathcal{E})$-scheme is the data of a morphism $g\colon T \to \mathbb{P}(\mathcal{E})$ and therefore the data of a sub-line bundle $\mathcal{L}_T \subset f^*\mathcal{E}$, if $f\colon T \to S$ denotes the composition of g with the projection to S. When we write T in what follows, we carry implicitly the above information. The points of U are then

$$U(T) = \{\mathcal{M} \subset f^*\mathcal{E} \oplus \mathcal{O}_T \mid \mathcal{M} \text{ a sub-l.b. and } \mathcal{M}_{|\mathcal{E}} = \mathcal{L}_T\}.$$

The points of $\mathbb{V}(\mathcal{O}(1))$ are

$$\mathbb{V}(\mathcal{O}(1))(T) = \mathcal{L}_T^\vee(T).$$

To see that these two functors are isomorphic, note that sending $\mathcal{M} \in U(T)$ to

$$\phi_{\mathcal{M}} \colon \mathcal{L} \subset \mathcal{M} \subset f^*\mathcal{E} \oplus \mathcal{O}_T \to \mathcal{O}_T$$

and $\phi \in \mathcal{L}_T^\vee(T)$ to the sub-line bundle generated by

$$\langle (v, \phi(v)) \rangle_{v \in \mathcal{L}_T} \in f^*\mathcal{E} \oplus \mathcal{O}_T$$

is an isomorphism.

□

Exercise 5. *Tautological line bundle.* This exercise is a direct follow-up to the preceding one. We call

$$\mathcal{O}(-1) \subset \pi^*\mathcal{E}$$

the *tautological line bundle.* We gather in this exercise various properties of this *universal* line bundle.
Say U is an affine of S, $M = \mathcal{E}(U)$ and $\varphi \in M^\vee$. Let $c \in M \otimes M^\vee$ be the canonical element (corresponding to the identity along the natural isomorphism $M \otimes M^\vee \cong \operatorname{Hom}_A(M, M)$).

(1) Show that $\mathcal{O}(-1)$ can be realized as the sub-line bundle of $\pi^*\mathcal{E}$ generated on $D_+(\varphi)$ by

$$c/\varphi \in \pi^* M(D_+(\varphi)) = \operatorname{Sym}(M^\vee)_{(\varphi)} \otimes M.$$

(2) Let $f \colon T \to S$ an S-scheme. From the previous exercise, deduce that if $T \to \mathbb{P}(\mathcal{E})$ is the map of S-schemes corresponding to an $\mathcal{L} \subset \pi^*\mathcal{E}$, then the following square

$$\begin{tikzcd} \mathbb{V}(\mathcal{L}) \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{O}(-1)) \arrow[d] \\ T \arrow[r] & \mathbb{P}(\mathcal{E}) \end{tikzcd}$$

is Cartesian.
Remark. The above says that $\mathbb{P}(\mathcal{E})$ is the *moduli space of sub-line bundles of* $\mathcal{E}$, and that $\mathcal{O}(-1)$ is the *universal line bundle on the moduli.*

(3) Show that $\mathbb{V}(\mathcal{O}(-1))$ is a closed subscheme of $\mathbb{V}(\pi^*\mathcal{E}) = \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$. This comes from the surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$.

(4) Let $f \colon T \to S$ an S-scheme. Show that a map of S-schemes $T \to \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$ which corresponds to a pair $(\mathcal{L}, v)$ with $\mathcal{L} \subset f^*\mathcal{E}$ and $v \in f^*\mathcal{E}(T)$ factors through $\mathbb{V}(\mathcal{O}(-1))$ if and only if $v \in \mathcal{L}(T)$.
Remark. In particular if $S = \operatorname{Spec}(k)$ where k is a field, and $\mathcal{E} = k^{n+1}$, the bundle $\mathbb{V}(\mathcal{O}(-1))$ is realized as a closed subscheme of $\mathbb{A}_k^{n+1} \times_k \mathbb{P}_k^n$.

Solution key. (1) A local claim suffices. So say $S = \operatorname{Spec}(A)$ is affine and $\mathcal{E}$ can be identified with a finite projective A-module M. The built-in surjection $\pi^* M^\vee \to \mathcal{O}(1)$ reads on $D_+(\varphi)$ as the surjective map

$$\alpha \colon \operatorname{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee \to \mathcal{O}(1)(D_+(\varphi))$$

determined by $1 \otimes \psi \mapsto \psi$. We want to dualize it to understand the claim. Namely we want to understand the image of the dual. Note that to determine that, we can post compose by an isomorphism the above map, so we can trivialize by $\frac{1}{\varphi}\colon \mathcal{O}(1)(D_+(\varphi)) \to \mathcal{O}(D_+(\phi))$. Therefore we want to analyze the dual of the map

$$\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee \to \mathrm{Sym}(M^\vee)_{(\varphi)}$$

determined by sending $1 \otimes \psi \to \frac{\psi}{\phi}$.

Recall that (because M is finite projective) we have an isomorphism

$$\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M \to \mathrm{Hom}_{\mathrm{Sym}(M^\vee)_{(\varphi)}}(\mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M^\vee, \mathrm{Sym}(M^\vee)_{(\varphi)})$$

via the map determined by sending $m \in M$ to the map determined by sending $\psi \in M^\vee$ to $\psi(m)$. The image of the dual of α on the right side is given, as explained above by the map determined by $1 \otimes \psi \mapsto \frac{\psi}{\varphi}$. So it suffice to check that $\frac{c}{\varphi}$ is sent to this map. Recall that if $c = \sum_i \phi_i \otimes m_i$, it has the property that for every $\psi \in M^\vee$ we have

$$\psi = \sum_i \psi(m_i)\phi_i.$$

Therefore the claim follows.

(2) By construction in the situation of the previous exercise $f^*\mathcal{O}(-1) = \mathcal{L}$. The claim follows.

(4) For the two last items, one notes that translating the factorization into the closed subscheme $\mathbb{V}(\mathcal{O}(-1))$ amounts to the existence of a factorization

$$\begin{tikzcd}
f^*\mathcal{E}^\vee \arrow[d] \arrow[dr] & \\
\mathcal{L}^\vee \arrow[r, dashed] & \mathcal{O}_T
\end{tikzcd}$$

which amounts by dualizing to the claim.

Remark. Yet another perspective on $\mathbb{V}(\mathcal{O}(-1))$ is that it is the *blow-up of* $\mathbb{V}(\mathcal{E})$ *at the zero section.*

□

Exercise 6. *Stability properties of (very-)ample sheaves under tensor product.* Let X be a Noetherian scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X.

(1) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, show that $\mathcal{L} \otimes \mathcal{M}$ is ample.
(2) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, deduce that there is a n such that $\mathcal{L}^n \otimes \mathcal{M}$ is ample.
(3) Show that if $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Now suppose that X is an A-scheme where A is a Noetherian ring.

(4) If $\mathcal{L}$ is A-very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is A-very ample.

(5) If $\mathcal{L}$ is ample, then there is a $n_0 > 0$ such that $\mathcal{L}^n$ is A-very-ample for all $n \geq n_0$.

Solution key. (1) First note that if $\mathcal{F}$ and $\mathcal{G}$ are globally generated and then $\mathcal{F} \otimes \mathcal{G}$ is also because all pure tensors of global sections $f \otimes g$ because this is a local claim and the tensor product of two surjective map is surjective. The claim follows.

(2) Follows.

(3) Same.

(4) Choose sections $s_0, \ldots, s_n \in \mathcal{L}(X)$ which defines an A-immersion $X \to \mathbb{P}^n_A$ and $m_0, \ldots, m_m \in \mathcal{M}(X)$ that defines an A-morphism $X \to \mathbb{P}^m_A$. Then the product $X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A$ is an immersion as immersions are closed under base-change. Now conclude using a Segre embedding.

(5) Follows from previous point and the proposition shown in class that there exists *some* n_0 with $\mathcal{L}^{n_0}$ being A-very ample.

□

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AG II - Schemes and sheaves

Solutions – week 14

Exercise 1. *A short exact sequence.* Let $\iota\colon D \to X$ be an effective Cartier divisor on a integral scheme X. Show that there is a short exact of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \iota_*\mathcal{O}_D \to 0.$$

Deduce that there is a long exact sequence in cohomology,

$$(\dots) \to \mathrm{H}^i(X, \mathcal{O}(-D)) \to \mathrm{H}^i(X, \mathcal{O}_X) \to \mathrm{H}^i(D, \mathcal{O}_D) \to (\dots)$$

Solution key. $\mathcal{O}(-D)$ is the ideal sheaf associated to the closed subscheme D.

☐

Exercise 2. Let $\iota : Z \to X$ be a closed immersion of schemes, where Z and X are not necessarily noetherian schemes.

(1) Show that the functors $R^j\iota_* : \mathrm{Mod}_{\mathcal{O}_Z} \to \mathrm{Mod}_{\mathcal{O}_X}$ are zero for all $j > 0$.
(2) Conclude that for an $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_Z}$, $\mathrm{H}^i(Z, \mathcal{F}) \cong \mathrm{H}^i(X, \iota_*\mathcal{F})$ for all $i \in \mathbb{N}$.

Solution key. The functor j_* is exact, which can be checked at stalks. The assertion on higher pushforwards follows.
Consider morphisms of ringed spaces (the right one is the terminal ringed space)

$$(Z, \mathcal{O}_Z) \xrightarrow{\iota} (X, \mathcal{O}_X) \xrightarrow{p} (*, \mathbb{Z}).$$

It holds in general that $R(p\circ\iota)_* = Rp_* \circ R\iota_*$ at the level of derived categories of $\mathcal{O}$-modules.
As the first point shows that $R\iota_* = \iota_*$, this concludes.

☐

Exercise 3. *A geometric perspective on the Euler sequence.* Let A be a ring and M a locally free of finite rank A-module.

(1) *Directional derivative.* For a $v \in M$, show that there is a unique A-derivation
$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee) \to \mathrm{Sym}(M^\vee)$$
which is equal to the evaluation at v on elements of degree 1. If M is free, if (e_i) and (x_i) denotes a basis and a dual basis respectively, and $v = \sum \lambda_i e_i$, show that
$$\frac{\partial}{\partial v} = \sum_i \lambda_i \frac{\partial}{\partial x_i}.$$

(2) For $\varphi \in M^\vee$, show that $\frac{\partial}{\partial v}$ uniquely extends to an A-derivation

$$\frac{\partial}{\partial v}\colon \operatorname{Sym}(M^\vee)_\varphi \to \operatorname{Sym}(M^\vee)_\varphi.$$

Deduce that $\frac{\partial}{\partial v}$ defines an A-derivation

$$\frac{\partial}{\partial v}\colon \operatorname{Sym}(M^\vee)_{(\varphi)} \to \operatorname{Sym}(M^\vee)(-1)_{(\varphi)}.$$

(3) Denote by $\pi\colon \mathbb{P}(M) \to \operatorname{Spec}(A)$ and $\mathcal{T}^1_{\mathbb{P}(M)|A} = \left(\Omega^1_{\mathbb{P}(M)|A}\right)^\vee$. Deduce that there is a $\mathcal{O}_{\mathbb{P}(M)}$-linear map

$$\frac{\partial}{\partial(-)}\colon \pi^*M \to \mathcal{T}^1_{\mathbb{P}(M)|A}(-1).$$

Hint: $\mathcal{T}^1_{\mathbb{P}(M)|A}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(M)}}(\Omega^1_{\mathbb{P}(M)|A}, \mathcal{O}(-1))$[1]. *Use the universal property of* $\Omega^1_{\mathbb{P}(M)|A}$ *on affines* $D_+(\varphi)$.

(4) *Euler sequence.* Let S be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank on S. Show that there is an exact sequence of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-locally free sheaves

$$0 \to \mathcal{O}(-1) \to \pi^*\mathcal{E} \xrightarrow{\frac{\partial}{\partial(-)}} \mathcal{T}^1_{\mathbb{P}(\mathcal{E})|S}(-1) \to 0$$

where the first arrow is the canonical inclusion $\mathcal{O}(-1) \to \pi^*\mathcal{E}$ and the second is a globalization of the arrow above. *Hint: Use the naturality of the construction to reduce to a case where the base is affine and* $\mathcal{E}$ *is free. We are now working in* $\mathbb{P}^n_A$. *Choose a basis and write the matrices of maps in question on opens* $D_+(x_i)$.

Solution key. (1) Because of the Leibniz rule, it suffice to determine an A-derivation on degree 1 elements.

(2) Note that because of the Leibniz rule, the following is forced.

$$0 = \varphi \frac{\partial}{\partial v}\frac{1}{\varphi} + \frac{1}{\varphi}\frac{\partial}{\partial v}(\varphi).$$

Therefore, we define

$$\frac{\partial}{\partial v}\frac{1}{\varphi} = -\frac{1}{\varphi^2}\frac{\partial}{\partial v}(\varphi) = -\frac{1}{\varphi^2}\varphi(v).$$

By the Leibniz rule, it extends. The second claim follows because such a derivation decreases the degree by 1.

(3) On an affine $D_+(\varphi)$ we define

$$M \otimes \operatorname{Sym}(M^\vee)_{(\varphi)} \to \operatorname{Der}_A(\operatorname{Sym}(M^\vee)_{(\varphi)}, \operatorname{Sym}(M^\vee)_{(\varphi)}(-1))$$

by sending $v \otimes \frac{f}{\varphi^{\deg(f)}}$ to the derivation $\frac{f}{\varphi^{\deg(f)}}\frac{\partial}{\partial v}$. This glues because this only depends on where to send M.

---

[1] Because in general if $\mathcal{F}$ is finite locally free and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$

(4) Working locally is enough. Indeed the sequence is functorial in $\mathcal{E}$, so this is sufficient. Let's say that $\mathcal{E} = A^{n+1}$. Fix an $i \in \{0, \dots, n\}$, without loss of generality say $i = 0$. Denote by $t_j = \frac{x_j}{x_0}$. Note that (we just need to choose where to send each t_j)

$$\mathrm{Der}_A(A[t_1, \dots, t_n], \frac{1}{x_0}A[t_1, \dots, t_n]) = \bigoplus_{j \geq 1} A[t_1, \dots, t_n]\frac{1}{x_0}.$$

In term of the above basis, we have $\frac{\partial}{\partial x_0} = (-\frac{x_1}{x_0}, \dots, -\frac{x_n}{x_0})$, because

$$\frac{\partial}{\partial x_0}(\frac{x_j}{x_0}) = \frac{-x_j}{x_0^2} = \frac{1}{x_0}\frac{-x_j}{x_0}.$$

Also we have $\frac{\partial}{\partial x_j} = e_j$. As $e_i \in (A[t_0, \dots, t_n])^{\oplus n+1}$ is sent to $\frac{\partial}{\partial x_i}$ we get that the martix of the map is

$$\begin{pmatrix} -\frac{x_1}{x_0} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_n}{x_0} & 0 & \cdots & 1 \end{pmatrix}.$$

It follows that the map is surjective and that the kernel is generated by

$$\sum_{j=0}^{n} e_j \otimes \frac{x_j}{x_0} = \frac{c}{x_0}$$

which proves the claim. (We used the notation c from the exercise on the tautological line bundle).

□

Exercise 4. *Cohomology and affine maps.* Let $X \to Y$ be an affine map of schemes and $\mathcal{F}$ a quasi-coherent sheaf on X.

(1) Show that the natural map $f_* \to Rf_*$ is an isomorphism, meaning that $R^i f_* = 0$ if $i > 0$.
(2) Deduce that

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(Y, f_*\mathcal{F}).$$

(3) Let E be again the curve from the above exercise. Let $f\colon E \to \mathbb{P}^1_{\mathbb{C}}$ be the restriction of the partially defined projection $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_C$ on the first two components. Show that this is well defined and compute the cohomology of $f_*\mathcal{O}(nP_0)$ for $n \in \mathbb{Z}$.

Solution key. Recall that $R^i f_*$ is the sheafification of $U \mapsto \mathrm{H}^i(f^{-1}(U), \mathcal{F})$. Because f is affine, the latter vanishes, the first claim therefore follows.

Consider morphisms of ringed spaces (the right one is the terminal ringed space)

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (*, \mathbb{Z}).$$

It holds in general that $R(g \circ f)_* = Rg_* \circ Rf_*$ at the level of derived categories of $\mathcal{O}$-modules.

The first point of the exercise shows that when $\mathcal{F}$ is quasi-coherent, then $Rf_*(\mathcal{F}) = f_*\mathcal{F}$. Using this and the above we get $R(g\circ f)_*\mathcal{F} = Rg_* \circ Rf_*\mathcal{F} = Rg_*(f_*\mathcal{F})$, which concludes.
If we work on Noetherian schemes, we can use that any injective quasi-coherent is flasque. By the previous point f_* sends an injective quasi-coherent resolution of $\mathcal{F}$, to a flasque *resolution* (by exactness on quasi-coherent sheaves) of $f_*\mathcal{F}$. The second claim also follows this way in this case.

□

Exercise 5. *Curves in* $\mathbb{P}^2_k$. Let k be a field. Let $C = V_+(F)$ for $F \in \mathcal{O}_{\mathbb{P}^2_k}(d)(\mathbb{P}^2_k)$ for a $d \geq 1$.

(1) Show that $\mathrm{H}^0(C, \mathcal{O}_C) \cong k$.
(2) Deduce that any C_1 and C_2 of the above form intersect.
(3) Suppose that C does not contain $[0 : 0 : 1]$ (this can always be arranged up to an automorphism of $\mathbb{P}^2_k$). Calculate the Čech complex associated to the cover $C \cap D_+(Y) \cup C \cap D_+(X)$ explicitly and deduce that $\mathrm{H}^1(C, \mathcal{O}_C)$ is a k-vector space of dimension $\frac{(d-1)(d-2)}{2}$.

Solution key. For the first item, use that $\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}(-d)) = 0$. For the second item, suppose by contradiction that two curves $V_+(F)$ and $V_+(G)$ do not intersect. Then the union is not connected. But the union is described as $V_+(FG)$. Being disconnected, they would be non trivial idempotents in the global sections, a contradiction. We could conduct a Čech cohomology computation to get the last result, but we indicate that it follows from the long exact sequence in cohomology of

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0.$$

□

Remark. We say that $\frac{(d-1)(d-2)}{2}$ is the *arithmetic genus* of C. Curves of degree 3 are of arithmetic genus 1. Smooth ones are called *elliptic curves.* Any smooth curve C over an algebraically closed field k with $\mathrm{H}^1(E, \mathcal{O}_E) = 1$ can be realized as a smooth cubic in $\mathbb{P}^2_k$, see for example Harthshorne III,4.6.

Exercise 6. *A Čech cohomology computation.* Let k be a field. Let $U = \mathbb{A}^2_k \setminus 0$. Compute the cohomology of $\mathcal{O}_U$ on U. After showing that $\mathcal{O}_U$ is ample, deduce that Serre vanishing does not hold for U.

Solution key. The Cech complex of $\mathcal{O}_U$ with respect to the open cover $\mathcal{U} = \{D(x), D(y)\}$ is

$$\begin{tikzcd}
C^0(\mathcal{O}_U) \arrow[r, "d"] \arrow[d, equal] & C^1(\mathcal{O}_U) \arrow[r] \arrow[d, equal] & \mathcal{C}^2(\mathcal{O}_U) \arrow[r] \arrow[d, equal] & \dots \\
\mathcal{O}_U(D(x)) \times \mathcal{O}_U(D(y)) \arrow[d, equal] & \mathcal{O}_U(D(xy)) \arrow[d, equal] & 0 & \\
k[x, x^{-1}, y] \times k[x, y, y^{-1}] & k[x, x^{-1}, y, y^{-1}] & & \\
(s,t) \arrow[r, mapsto] & s-t & &
\end{tikzcd}$$

Thus we have

$$\check{H}^1(\mathcal{U}, \mathcal{O}_U) = \frac{k[x, x^{-1}, y, y^{-1}]}{\mathrm{Im}(d)} = \frac{k[x, x^{-1}, y, y^{-1}]}{k[x, x^{-1}, y] + k[x, y, y^{-1}]} = \bigoplus_{i,j<0} \langle x^i y^j \rangle.$$

Note that $\mathcal{O}_U$ is k-very-ample. Indeed viewing U in $\mathbb{P}^2_k$, the pullback of $\mathcal{O}(1)$ on U is trivial. It now follows that $\mathcal{O}_U$ is k-very ample.

□

Exercise 7. *Coherence of derived pushforward:* Let $f : X \to Y$ be a projective morphism between two noetherian schemes. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, show that $R^i f_* \mathcal{F}$ is coherent for all i.

Solution key. Recall that if $j\colon (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an inclusion of an open, then j^* sends injective $\mathcal{O}_X$-modules to $\mathcal{O}_U$-modules because it admits $j_!$ is an exact left adjoint. Therefore if $U = \mathrm{Spec}(A) \subset Y$ is open, if we consider the pullback

$$\begin{tikzcd}
U' \arrow[r, "j'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
U \arrow[r, "j"] & Y
\end{tikzcd}$$

then we have

$$(R^i f_* \mathcal{F})_{|U} = (R^i f'_* \mathcal{F}_{|U'}).$$

So the conclusion follows from the affine case which was seen in class as a consequence of the computation of the cohomology of line bundles $\mathcal{O}(n)$ on $\mathbb{P}^n_A$ and of the theory of ample invertible sheaves.

□