

The questions are independent of each other in the sense that one can admit previous answers to treat the next questions. One need not solve all questions to get the maximal grade, and questions can be solved in any order. Good luck! Bonne chance! Viel Erfolg!

Exercise 1 (Calderón's theorem). In this exercise, we will prove Calderón's following result.

Theorem. Let $d \geq 2$, and $\Omega \subset \mathbb{R}^d$ be a bounded open connected domain. Then, the vector space spanned by products of real-valued harmonic functions is dense in $L^2(\Omega, \mathbb{R})$.

Recall that a real-valued function $u \in C^\infty(\overline{\Omega}, \mathbb{R})$ is said harmonic if $\Delta u = 0$, where

$$\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian operator.

We let $\mathcal{H} = C^\infty(\overline{\Omega}, \mathbb{R}) \cap \{u : \Delta u = 0 \text{ in } \Omega\}$, $\Pi = \text{Span} \{u v : (u, v) \in \mathcal{H} \times \mathcal{H}\}$.

(1) Let

$$A = \mathbb{C}^d \cap \left\{ \zeta : \zeta \cdot \zeta = \sum_{j=1}^d \zeta_j^2 = 0 \right\}.$$

Show that for all $\zeta \in A$, the function $u : \Omega \rightarrow \mathbb{C}, u(x) = e^{i x \cdot \zeta}$ is harmonic.

(2) Fix some $f \in L^2(\Omega, \mathbb{R})$, and assume that

$$\int_{\Omega} f w \, dx = 0 \quad \text{for all } w \in \Pi.$$

Using well-chosen functions defined in (1), show that $\mathcal{F}(f \mathbf{1}_{\Omega}) = 0$, where $\mathbf{1}_{\Omega}$ is the indicator function of Ω , and \mathcal{F} is the Fourier transform. What can you deduce about f ?

(3) Conclude the proof of the theorem.

Exercise 2 (Basic estimates for the Schrödinger equation). (1) Let $d \geq 1$, and $\alpha > 0$ be a fixed real number. Using the formula

$$\mathcal{F} \left(\mathbb{R}^d \rightarrow \mathbb{R}, x \mapsto e^{-\alpha x^2} \right) (\xi) = \left(\frac{\pi}{\alpha} \right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\alpha}}$$

for all $\xi \in \mathbb{R}^n$, show that the function $g_{\alpha} : \mathbb{R}^n \rightarrow \mathbb{C}, x \mapsto e^{-i\alpha|x|^2}$ is a tempered distribution (an element of $\mathcal{S}'(\mathbb{R}^d)$), and show that its Fourier transform is given by

$$\widehat{g}_{\alpha}(\xi) = \left(\frac{\pi}{i\alpha} \right)^{\frac{d}{2}} e^{i\frac{|\xi|^2}{4\alpha}}.$$

(2) Let $f \in \mathcal{S}(\mathbb{R}^d)$, and consider the following partial differential equation:

$$\begin{cases} i \frac{\partial}{\partial t} u + \Delta u = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = f(x) \end{cases}, \quad (1)$$

where we recall that the Laplacian Δ is defined by

$$\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2},$$

and $u \in C^1((0, \infty), \mathcal{S}'(\mathbb{R}^d))$. Using the previous question, show that the following representation formula holds for u :

$$u(t, x) = \frac{1}{(2\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y) dy \quad \text{for all } (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d.$$

(3) Show that $u \in C^\infty((0, \infty) \times \mathbb{R}^d)$, and that for all $t > 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |u(t, x)| \leq \frac{1}{(2\pi t)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)},$$

and that for all $t > 0$, we have

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|u(0, \cdot)\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

More generally, estimate for all $s \in \mathbb{R}$ the following norm $\|u(t, \cdot)\|_{H^s(\mathbb{R}^d)}$ ($t > 0$) in terms of $\|f\|_{H^s(\mathbb{R}^d)}$.

(4) Let $2 \leq p \leq \infty$. By a scaling argument, show that provided that the inequality

$$\|u(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{t^\alpha} \|f\|_{L^{p'}(\mathbb{R}^d)}$$

holds for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all solution u of (1) (with initial data f), then $\alpha = d \left(\frac{1}{2} - \frac{1}{p} \right)$. One may introduce the following function $u_\lambda(t, x) = u(\lambda^2 t, \lambda x)$ for $\lambda > 0$ in the proof.

This estimate is one of the basic ingredients of the proof of the space-time Strichartz estimates:

$$\|u\|_{L^p(\mathbb{R}_+, L^q(\mathbb{R}^d))} \leq C(p, q, d) \|f\|_{L^2(\mathbb{R}^d)}$$

for all $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $(p, q) \neq (2, \infty)$, $p \geq 2$.

Exercise 3 (Elliptic Regularity). This exercise aims at generalising the *elliptic estimates* on the harmonic functions. Namely, $\Delta u = f \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$. Here, we will prove the main step that shows that weak solution of elliptic partial differential equation are continuous.

Let $\Omega \subset \mathbb{R}^d$ be a connected open subset. Recall that for all $1 \leq p, q \leq \infty$, for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\varphi \in W_0^{1,q}(\Omega)$ such that $u \nabla \varphi \in L^1(\Omega)$ and $\varphi \nabla u \in L^1(\Omega)$, we have for all $1 \leq j \leq d$

$$\int_{\Omega} u \partial_{x_j} \varphi \, dx = - \int_{\Omega} (\partial_{x_j} u) \varphi \, dx.$$

Restricting to $\Omega = B(0, 1)$ from now on, we let $A = (a_{i,j})_{1 \leq j \leq d} \in L^\infty(B(0, 1), \mathbb{R}^d)$ be a space-dependent *uniformly elliptic* matrix, *i.e.* there exists $0 < \Lambda < \infty$ such that

$$\Lambda^{-1} |\xi|^2 \leq \xi^t A(x) \xi = \sum_{i,j=1}^d a_{i,j}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall x \in B(0, 1), \forall \xi \in \mathbb{R}^d. \quad (1)$$

We now let $u \in W^{1,2}(B(0, 1))$ be a weak solution (in $\mathcal{D}'(B(0, 1))$) of the linear partial differential equation

$$\operatorname{div}(A(x) \nabla u) = 0. \quad (2)$$

Important note: all inequalities can be proven with “worse” constants than stated, provided that they are universal constants (only depend on the ambient dimension and the parameter Λ), unless stated otherwise.

(1) For $d = 1$, show directly that $u \in C^0(B(0, 1))$.

(2) Show that for all $\varphi \in \mathcal{D}(B(0, 1)) = C_c^\infty(B(0, 1))$, the following identity holds

$$\int_{B(0,1)} A(x) \nabla u \cdot \nabla \varphi \, dx = 0.$$

Notice that for all $u, v \in H^1(B(0, 1))$, we have

$$\int_{B(0,1)} A(x) \nabla u \cdot \nabla v \, dx = \sum_{i,j=1}^d \int_{B(0,1)} a_{i,j}(x) \partial_{x_i} u \partial_{x_j} v \, dx.$$

- (3) (Caccioppoli inequality) Let $u \in W^{1,2}(B(0,1))$. Show that for all $\eta \in \mathcal{D}(B(0,1))$, and $c \in \mathbb{R}$, we have

$$\int_{B(0,1)} |\nabla u|^2 \eta^2 dx \leq \Lambda^2 \int_{B(0,1)} |u - c|^2 |\nabla \eta|^2 dx.$$

One can use the following test function $\varphi = (u - c)\eta^2$.

- (4) By choosing an appropriate test function η , show that for all $0 < r_1 < r_2 < 1$, we have for some universal constant $0 < C < \infty$,

$$\int_{B(0,r_1)} |\nabla u|^2 dx \leq \frac{C}{(r_2 - r_1)^2} \int_{B(0,r_2) \setminus \overline{B(0,r_1)}} |u - c|^2 dx \quad (3)$$

This is the *Caccioppoli inequality*.

- (5) Using the Poincaré-Wirtinger inequality and a scaling argument, show that for all $0 < r < \infty$ if $A(r) = B(0, 2r) \setminus \overline{B(0, r)}$ and $u \in W^{1,2}(B(0, r))$, we have

$$\int_{A(r)} |u - u_r|^2 dx \leq Cr^2 \int_{A(r)} |\nabla u|^2 dx,$$

where $u_r = \oint_{A(r)} u dx = \frac{1}{c_d r^d} \int_{A(r)} u dx$ (for some universal constant c_d) is the mean of u on $A(r)$, and $0 < C < \infty$ is a universal constant independent of r .

- (6) Deduce from the previous questions that there exists a universal constant $0 < \theta < 1$ such that for all $0 < r < \frac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \theta \int_{B(0,2r)} |\nabla u|^2 dx.$$

By induction, deduce that there exists $0 < \alpha < 1$ and $0 < C < \infty$ such that for all $0 < r < \frac{1}{2}$, we have

$$\int_{B(0,r)} |\nabla u|^2 \leq Cr^\alpha \int_{B(0,1)} |\nabla u|^2 dx.$$

Hint: use a dyadic argument.

- (7) With the help of another optimal Poincaré-Wirtinger inequality, deduce that there exists a universal constant $0 < C < \infty$ such that for all $0 < r < \frac{1}{2}$,

$$\int_{B(0,r)} |u - \tilde{u}_r|^2 dx \leq Cr^{2+\alpha} \int_{B(0,1)} |\nabla u|^2 dx,$$

where $\tilde{u}_r = \oint_{B(0,r)} u dx = \frac{1}{c_d r^d} \int_{B(0,r)} u dx$ is the mean of u on $B(0, r)$.

- (8*) In the special case $d = 2$, show with the help of Lebesgue differentiation theorem and a translation argument that $u \in C^{0,\beta}(B(0, \frac{1}{4}))$ for some $0 < \beta < 1$. Recall that the Lebesgue differentiation theorem shows that for all $u \in L^1_{\text{loc}}(B(0, 1))$, and for all $x \in B(0, 1)$, we have*

$$\oint_{B(x,r)} u(y) dy = \frac{1}{\pi r^2} \int_{B(x,r)} u(y) dy \xrightarrow{r \rightarrow 0} u(x).$$

*Notice that the theorem is stated on \mathbb{R}^2 .

Furthermore, recall that the space of β -Hölder functions is defined by

$$C^{0,\beta}(\Omega) = C^0(\Omega) \cap \left\{ u : \|u\|_{C^{0,\beta}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty \right\}$$

and equipped with the $\|u\|_{C^{0,\beta}(\Omega)}$ norm defined above. One may first prove the following estimate

$$|u_{x_0,r} - u_{x_0,s}| \leq C \frac{r^{1+\frac{\alpha}{2}}}{s} \int_{B(0,1)} |\nabla u|^2 dx$$

for all $x_0 \in B(0, \frac{1}{2})$, and $0 < s < r < \frac{1}{2}$, where $u_{x_0,r} = \oint_{B(x_0,r)} u dx = \frac{1}{\pi r^2} \int_{B(x_0,r)} u dx$.

- (9) Assume from now on that $d \geq 3$. We say that $u \in H^1(B(0,1))$ is a weak sub-solution if for all $\varphi \in \mathcal{D}(B(0,1))$, we have

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \varphi dx \leq 0, \quad (4)$$

or in other words

$$\sum_{i,j=1}^d \int_{\Omega} a_{i,j}(x) \partial_{x_i} u \partial_{x_j} v dx \leq 0.$$

Using the chain rule and the product formula for Sobolev functions, show that for all convex, increasing function $\Phi \in C^2(\mathbb{R}, \mathbb{R}_+)$ such that $\Phi''(t) = 0$ for all $|t| \geq R$ (for some fixed $R > 0$), then for any sub-solution $u \in H^1(B(0,1))$, the function $\Phi(u) \in H^1(B(0,1))$ is a sub-solution.

From now on, we assume that the previous result holds for any increasing, convex function $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ (not necessarily C^2 or satisfying the extra hypothesis on the second derivative).

- (10) (Moser iterations) Fix some $0 < 2r < 1$, and for all $j \in \mathbb{N}$, let $B_j = B(0, r + 2^{-j}r)$, so that $B_{j+1} \subset B_j \subset B(0, 2r)$ and

$$B_\infty = \bigcap_{j \in \mathbb{N}} B_j = B(0, r).$$

Show that for all $1 \leq \lambda \leq \frac{d}{d-2}$, there exists $0 < \gamma < \infty$ such that for all $j \in \mathbb{N}$, for all $v \in H^1(B_{j+1})$, we have

$$\| |v|^\lambda \|^2_{L^2(B_{j+1})} \leq \gamma \|\nabla v\|_{L^2(B_{j+1})}^{2\lambda} + \gamma \|v\|_{L^2(B_{j+1})}^{2\lambda}.$$

- (11) Fix $1 < \lambda \leq \frac{d}{d-2}$, and let $u \in H^1(B_j)$ a *positive* weak sub-solution. With the help of the Caccioppoli inequality (explain why it holds true for a sub-solution), show that $u^\lambda \in H^1(B_{j+1})$, that u^λ is a positive weak sub-solution, and show the inequality

$$\|u^\lambda\|_{L^2(B_{j+1})}^2 \leq C(2^{2\lambda j} + 1) \|u\|_{L^2(B_j)}^{2\lambda},$$

where C only depends on $d, \Lambda, r > 0$.

- (12) Let $u \in H^1(B(0,1))$ be a positive weak sub-solution. Define for all $j \in \mathbb{N}$ the function $u_j = u^{\lambda^j}$. Explain why

$$\|u\|_{L^\infty(B(0,r))} = \limsup_{j \rightarrow \infty} \|u\|_{L^{2\lambda^j}(B_j)},$$

and prove the following inequality

$$\|u\|_{L^\infty(B(0,r))} \leq C(d, \Lambda, r) \|u\|_{L^2(B(0,2r))}.$$