Distribution & Interpolation Spaces – Solution sheet 2

Exercise 1. For the convenience of the readability and without loss of generality we can suppose that f is non-negative, namely $f : X \to [0, \infty]$. The exercise is an application of Tonelli theorem. Denote by

$$X_f := \{(x,t) \in X \times [0, +\infty) : t < f(x)\}$$

the hypograph of the function f and observe that for every $x \in X$ it holds $\int_0^{f(x)} 1dt = f(x)$. Therefore

$$\int_X f(x)dx = \int_X \left(\int_0^{f(x)} 1dt\right) dx = \int_X \left(\int_0^{+\infty} \chi_{[0,f(x)]}(t)dt\right) dx = \int_{X\times[0,+\infty)} \chi_{X_f} dtdx,$$

where in the last equality we used Tonelli theorem. R ecall that in Tonelli theorem we have the assumption μ be a σ finite measure. Again by Tonelli theorem,

$$\int_{X\times[0,+\infty)} \chi_{X_f} dtdx = \int_0^{+\infty} \left(\int_X \chi_{X_f}(t,x)dx\right) dt = \int_0^{+\infty} |\{x \in X : f(x) > t\}| \, dt.$$

This concludes the first part of the exercise, the second part follows the same computations with the only modification that

$$\int_0^{f(x)} pt^{p-1}dt = f(x)^p.$$

For the third part we notice that we cannot use Tonelli theorem. So we argue in the following way, if $f(x) = a\chi_A$ for some $a \in [0, \infty]$ and $A \subset X$ measurable, then the identity is straightforward. Let us suppose that $f(x) = \sum_{j=1}^N a_i \chi_{A_j}$ for some $a_j \geq 0$ and A_j measurable, then there exists $\{b_i\}_{i=1}^n$ such that $0 \leq b_1 < b_2 < ... < b_n$ and measurable sets $\{B_i\}_i$ such that

$$f(x) = \sum_{i=1}^n b_i \chi_{B_i},$$

therefore we have

$$\int_X f(x)d\mu(x) = \sum_{i=1}^n b_i \mu(B_i)$$

On the other hand (defining $b_0 = 0$)

$$\int_0^\infty \mu(x : \sum_{j=1}^n b_j \chi_{B_j}(x) > t)dt = \sum_{i=1}^n \int_{b_{i-1}}^{b_i} \mu(x : \sum_{j=1}^n b_j \chi_{B_j}(x) > t)dt$$
$$= \sum_{i=1}^n \sum_{j=i}^n \mu(B_j)(b_i - b_{i-1}) = \sum_{j=1}^n \sum_{i=1}^j (b_i - b_{i-1})\mu(B_j) = \sum_{j=1}^n b_j \mu(B_j).$$

Using monotone convergence theorem we can prove that the identity is true for any function f non-negative.

Exercise 2. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. Since $f \in L^1(\mathbb{R}^n)$, then we know that $f * \rho_\epsilon \to f$ in $L^1(\mathbb{R}^n)$ as $\epsilon \to 0$. Thus, up to subsequences, we can assume that

$$(f * \rho_\epsilon)(x) \to f(x) \quad \textit{for a.e. } x \in \mathbb{R}^n, \ \textit{as } \epsilon \to 0\,. \tag{1}$$

By hypothesis we have that, for any $\epsilon > 0$,

$$(f * \rho_\epsilon)(x) = \int_{\mathbb{R}^n} f(y)\rho_\epsilon(x-y)\,dy = 0\,, \tag{2}$$

since $\rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$. Putting together (1) and (2) we conclude that $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

Exercise 3 Let us fix a sequence $\{x_n\}_n$ such that $\|x_n - x\|_X \to 0$. We estimate

$$\|T_n(x_n) - T(x)\|_Y \leq \|T_n(x_n) - T_n(x)\|_Y + \|T_n(x) - T(x)\|_Y$$

for the second term we observe that

$$\|T_n(x) - T(x)\|_Y \to 0$$

by assumption, whereas we estimate the first term as

$$\|T_n(x_n) - T_n(x)\|_Y \leq \|T_n\|_{\mathcal{L}(X,Y)}\|x_n - x\|_X.$$

Using that $T_n(x) \to T(x)$ for all $x \in X$ and $T \in \mathcal{L}(X,Y)$ we get by Banach Steinhaus that

$$\sup_n \|T_n\|_{\mathcal{L}(X,Y)} < \infty$$

and therefore we conclude that

$$\|T_n\|_{\mathcal{L}(X,Y)}\|x_n - x\|_X \to 0$$

as $n \to \infty$.

Exercise 4 Let us define

$$T : X \to Y^*$$

as $T(x) = a(x,\cdot) \in \mathcal{L}(Y,\mathbb{R})$. The goal is to show that T is a bounded operator indeed

$$\sup_{\|y\|_Y \leq 1} |a(x,y)| = \sup_{\|y\|_Y \leq 1} < T(x), y >_{Y^*,Y} \leq \|T(x)\|_{Y^*} \leq \|T\|_{\mathcal{L}(X,Y^*)}\|x\|_X$$

for all $x \in X$, from which the thesis follows. Using Corollary 1.2.4 it is sufficient to fix $y \in Y$ and prove that

$$\sup_{\|x\|_X \leq 1} < T(x), y >_{Y^*,Y} \leq C$$

and this follows from the linearity and continuity in x of the bilinear operator $a(x,y)$.

Exercise 5 For every $x \in \ell^p$ set $T_n(x) = \sum_{i=1}^n a_i x_i$, so that $T_n(x) \to T(x) = \sum_{i=1}^\infty a_i x_i$ for avery $x \in \ell^p$. Therefore, applying Banach Steinhaus, if follows that there exists a constant $C > 0$ such that

$$\sup_n |T_n(x)| \leq C\|x\|_{\ell^p}$$

for any $x \in \ell^p$. Therefore (if $p > 1$) choosing $x = \{x_n\}_n$ such that $x_n = a_n^{p'-1}$ we get that

$$\|a\|_{\ell^{p'}}^{p'} = \sup_n |T_n(x)| \le C\|a\|_{\ell^{p'}}^{p'/p}$$

conclusing that $\|a\|_{\ell^{p'}} \le C$.

Exercise 6 (1) implies (2) because χ_A is an $L^{p'}$ function for any $A \subset \Omega$ of finite measure. The same for $p = 1$. To see that the sequence is bounded you can use Proposition 1.3.3.
(2) implies (1) because the vector space spanned by characteristic function χ_A with $A \subset \Omega$ of finite measure are dense in $L^{p'}$ (this is not true if $p' = \infty$ and the space $\mathcal{L}^d(\Omega) = \infty$).

Exercise 7 Suppose f is continuous and let U be a neighbourhood of $f(x)$. By definition of neighbourhood there is an open set $A \subset Y$ such that $f(x) \in A \subset U$: the open set $V = f^{-1}(A)$ is a neighbourhood of x and $f(V) \subset U$.
Conversely, suppose f is continuous at every point and let A be an open set in Y. If $x \in f^{-1}(A)$ then A is a neighbourhood of $f(x)$ and there is a neighbourhood V_x of x such that $f(V_x) \subset A$. This amounts to saying $V_x \subset f^{-1}(A)$. By definition there exists U_x open such that $x \in U_x$ and $U_x \subset V_x$. Therefore $f^{-1}(A)$ is an open set because it is union of open sets

$$f^{-1}(A) = \bigcup_{x \in f^{-1}(A)} U_x.$$