Distribution & Interpolation Spaces – Solution sheet 6

Exercise 1. We introduce the following notation: for any $\varphi \in C^\infty(\mathbb{R}^n)$ we define $\overline{\varphi}(x) = \varphi(-x)$ for any $x \in \mathbb{R}^n$. The convolution $(T * \varphi)(x) = \langle T, \tau_x \overline{\varphi} \rangle \in C^\infty(\mathbb{R}^n)$ and moreover $\left(D^\alpha(T * \varphi)\right)(x) = (T * D^\alpha \varphi)(x)$ for any multi-index α.

Let now $x \notin \operatorname{spt} T + \operatorname{spt} \varphi$. Then $\tau_x \overline{\varphi} = \overline{\varphi}(y - x) = \varphi(x - y) \in C_c^\infty(x - \operatorname{spt} \varphi)$ and since $x \notin \operatorname{spt} T + \operatorname{spt} \varphi$ we get that $x - \operatorname{spt} \varphi \cap \operatorname{spt} T = \emptyset$, from which we get

$$(T * \varphi)(x) = \langle T, \tau_x \overline{\varphi} \rangle = 0.$$

This shows that $\operatorname{spt}(T * \varphi) \subset \operatorname{spt} T + \operatorname{spt} \varphi$.

Exercise 2. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be arbitrary fixed. Recalling the notation used in the previous exercise, we compute

$$\langle T, \varphi \rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} = \langle T, \overline{\overline{\varphi}} \rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} = T * \overline{\varphi}(0) = S * \overline{\varphi}(0) = \langle S, \overline{\overline{\varphi}} \rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} = \langle S, \varphi \rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}}.$$

Exercise 3. We first recall the *Hint:* $T \in \mathcal{D}'(\Omega)$ and $f \in \mathcal{D}(\Omega \times \Omega)$ then

$$\left\langle T, \int_\Omega f(\cdot, y)\, dy \right\rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} = \int_\Omega \langle T, f(\cdot, y) \rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}}\, dy.$$

Thus we have

$$\begin{aligned}(T * (\varphi * \psi))(x) &= \left\langle T, \tau_x(\overline{\varphi * \psi}) \right\rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} = \left\langle T, \int_{\mathbb{R}^n} \psi(z) \varphi(x - \cdot - z)\, dz \right\rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} \\ &= \int_{\mathbb{R}^n} \psi(z) \left\langle T, \tau_{x-z} \overline{\varphi} \right\rangle_{\mathcal{D}' \leftrightarrow \mathcal{D}} dz = \int_{\mathbb{R}^n} \psi(z) T * \varphi(x - z) dz = ((T * \varphi) * \psi)(x).\end{aligned}$$

Thanks to the commutativity and associativity of convolution between distributions and smooth functions we directly get

$$(\delta * T) * \varphi = (T * \delta) * \varphi = T * (\delta * \varphi) = T * \varphi$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then the conclusion follows by applying the Exercise 2.

Exercise 4. It is sufficient to show that: for any $x_0 \in \Omega$ it holds that $u\varphi \in C^\infty(B_{\varepsilon/2})$, where $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ in $B_\varepsilon(x_0)$ for some $\varepsilon > 0$.

We notice that from exercise 2 of last exercise sheet and standard properties of convolutions (observe that $u\varphi \in \mathcal{E}'$) we have

$$u\varphi = u\varphi \star \delta_0 = u\varphi \star \Delta E = E \star \Delta(u\varphi)\,.$$

Finally, we take $\psi \in C_c^\infty(B_{\varepsilon/2}(0))$ and $\psi \equiv 1$ in $B_{\varepsilon/4}(0)$ to rewrite

$$E \star \Delta(u\varphi) = E\psi \star \Delta(u\varphi) + E(1-\psi) \star \Delta(u\varphi)\,.$$

We observe that the second piece is a smooth function because it is the convolution of a smooth function with a distribution with compact support. The first piece is such that

$$\mathrm{supp}(E\psi \star \Delta(u\varphi)) \subset \mathrm{supp}(E\psi) + \mathrm{supp}(\Delta(u\varphi)) \subset B_{\varepsilon/2}(0) + B_\varepsilon(x_0)^c \subset B_{\varepsilon/2}(x_0)^c$$

from which we get that the second piece is smooth in $B_{\varepsilon/2}(x_0)$ concluding the proof.

Exercise 5.
From the hypothesis it follows that u is also bounded. Furthermore we observe that also $\partial_{x_i} u$ is also an harmonic function which is smooth, therefore applying the mean value theorem (integrating also in dr from 0 to r_0) to $\partial_{x_i} u$ we get

$$\partial_{x_i} u(x_0)\mathcal{L}^d(B_1) r_0^d = \int_{B_{r_0}(x_0)} \partial_{x_i} u dx = \int_{\partial B_{r_)}(x_0)} u\nu_i \; d\mathcal{H}^{d-1},$$

where ν_i is the i-th component of the normal vector to $\partial B_{r_0}(x_0)$, therefore

$$|\partial_{x_i} u(x_0)| \leq (\mathcal{L}^d(B_1))^{-1} r_0^{-n} \|u\|_{L^\infty(\mathbb{R}^n)} \mathcal{H}^{d-1}(\partial B_{r_0}(x_0))$$

and the last goes to 0 as $r_0 \to \infty$. Therefore u is a constant because it has zero derivatives and then it is the constant zero by the assumption.

Exercise 6.
For any $f \in \mathcal{E}'$, using standard property of the convolution and the exercise 2 of the previous week we have that $u = G_d \star f$ is a solution of $\Delta u = f$ becaus

$$\Delta(G_d \star f) = \Delta G_d \star f = \delta_0 \star f = f\,.$$

The family $\{u_c\}_{c\in\mathbb{R}}$ defined as $u_c = u + c \in \mathcal{D}'$ is an uncountable family of solutions to the problem $\Delta u_c = f$.
We observe that all the solutions to $\Delta u = f$ are smooth outside the support of f because the support of f is a closed set so the complement $\Omega = \mathrm{supp}(f)^c$ is an open set and u on Ω solves $\Delta u = 0$, therefore using the exercise 4 we have that $u \in C^\infty(\Omega)$.
Finally, suppose that there are two solutions $u_1, u_2 \in \mathcal{D}'$ of

$$\Delta u = f$$

such that $u(x) \to 0$ as $|x| \to \infty$, then $w = u_1 - u_2 \in \mathcal{D}'$ solves

$$\Delta w = 0$$

and it is such that $w(x) \to 0$ as $|x| \to \infty$. Using exercise 4 we have $w \in C^\infty$ and using the exercise 5 we have $w \equiv 0$.