# Distribution & Interpolation Spaces – Solution sheet 5

**Exercise 1.**

*i)* It is sufficient to prove that $E \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Since $E \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, then it is enough to check that

$$\int_{B_1(0)} |E(x)| dx < \infty.$$

Indeed we can compute in polar coordinates

$$\frac{1}{2\pi}\int_{B_1(0)} |\mathrm{Log}\,|x||\, dx = \int_0^1 |r\,\mathrm{Log}(r)|\, dr = \frac{1}{2}.$$

*ii)* For every $x \neq 0$, we have

$$\nabla E(x) = \frac{1}{2\pi}\frac{x}{|x|^2}$$

so, for every $x \neq 0$,

$$\mathrm{div}\, \nabla E(x) = \frac{x}{2\pi}\cdot\nabla\frac{1}{|x|^2} + \frac{1}{2\pi|x|^2}\mathrm{div} x = \frac{x}{2\pi}\cdot\left(-\frac{2}{|x|^3}\right) + \frac{2}{2\pi|x|^2} = 0.$$

**Exercise 2.** By dominated convergence theorem we have

$$\int_{\mathbb{R}^2} E(x)\Delta\phi(x) dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} E(x)\Delta\phi(x) dx$$

Using twice the divergence theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} E\Delta\phi &= \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} E\nabla\phi\cdot\nu - \int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} \nabla E\cdot\nabla\phi \\ &= \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} E\nabla\phi\cdot\nu - \int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} \nabla E\cdot\nabla\phi \pm \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} \phi\nabla E\cdot\nu \qquad (1) \\ &= \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} E\nabla\phi\cdot\nu + \int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} \phi\Delta E + \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} \phi\nabla E\cdot\nu,\end{aligned}$$

where $\nu$ denotes the outer normal to the set $\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}$. We now consider the three terms of the last line separately. The second one is zero thanks to Point ii) of Exercise 1. We estimate the first term: since $|E(x)| = |\mathrm{Log}(\varepsilon)|$ for every $x \in \partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})$, and the measure of $\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})$ is $2\pi\varepsilon$, then

$$\left|\int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} E\nabla\phi\cdot\nu\right| \leq 2\pi\varepsilon\,\mathrm{Log}(\varepsilon)\,\mathrm{Sup}\,|\nabla\phi|.$$

In particular

$$\lim_{\varepsilon\to 0}\left|\int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})} E\nabla\phi\cdot\nu\right| = 0.$$

Now we consider the third term: for every $x\in\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})$ it holds $\nabla E(x)\cdot\nu(x)=\frac{1}{2\pi\varepsilon}$ and $|\phi(x)-\phi(0)|\leq\varepsilon\,\mathrm{Sup}\,|\nabla\phi|$, therefore

$$\begin{aligned}\int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}\phi\nabla E\cdot\nu &= \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}\phi(0)\nabla E\cdot\nu + \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}(\phi-\phi(0))\nabla E\cdot\nu\\ &= \phi(0) + \int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}(\phi-\phi(0))\nabla E\cdot\nu.\end{aligned}$$

Taking into account the estimates above and the fact that the measure of $\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})$ is $2\pi\varepsilon$, we can estimate

$$\left|\int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}(\phi-\phi(0))\nabla E\cdot\nu\right| \leq \mathrm{Sup}\,|\nabla\phi|\varepsilon$$

and in particular we obtain

$$\lim_{\varepsilon\to 0}\int_{\partial(\mathbb{R}^2\setminus\overline{B_\varepsilon(0)})}\phi\nabla E\cdot\nu = \phi(0).$$

In conclusion we have proved

$$\int_{\mathbb{R}^2} E(x)\Delta\phi(x)dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus\overline{B_\varepsilon(0)}} E(x)\Delta\phi(x)dx = \phi(0).$$

In order to prove that $\Delta E=\delta$, we observe that

$$\begin{aligned}\langle\Delta E,\phi\rangle &= \langle\partial_{11}E+\partial_{22}E,\phi\rangle = \langle\partial_{11}E,\phi\rangle+\langle\partial_{22}E,\phi\rangle = -\langle\partial_1E,\partial_1\phi\rangle-\langle\partial_2E,\partial_2\phi\rangle\\ &= \langle E,\partial_{11}\phi\rangle+\langle E,\partial_{22}\phi\rangle = \langle E,\Delta\phi\rangle.\end{aligned}$$

Since we proved $\langle E,\Delta\phi\rangle=\int_{\mathbb{R}^2}E(x)\Delta\phi(x)dx=\phi(0)$, this shows that $\langle\Delta E,\phi\rangle=\phi(0)$ for every $\phi\in\mathcal{D}(\mathbb{R}^2)$, namely $\Delta E=\delta$.

**Exercise 3.** Observe that the measurable function

$$f(x)=\begin{cases}\frac{1}{x} & \text{if } x>0,\\ 0 & \text{otherwise}\end{cases}$$

does *not* define a distribution on $\mathbb{R}$, since it does not belong to $L^1_{\mathrm{loc}}(\mathbb{R})$. On the other hand,

$$g(x)=\begin{cases}\mathrm{Log}(x) & \text{if } x>0,\\ 0 & \text{otherwise}\end{cases}$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ so it defines a distribution $T_g$ through

$$\langle T_g,\varphi\rangle = \int_{\mathbb{R}} g\varphi.$$

The distribution $T_g'$ satisfies the requirements of the exercise.

**Exercise 4.** Integrating by parts, we get

$$\begin{aligned} T_n(\varphi) &= \int_{-\infty}^{-\varepsilon_n^-} \frac{\varphi(x)}{x}dx + \int_{\varepsilon_n^-}^{+\infty} \frac{\varphi(x)}{x}dx - \int_{\varepsilon_n^-}^{\varepsilon_n^+} \frac{\varphi(x)}{x}dx \\ &= \int_{-\infty}^{-\varepsilon_n^-} \frac{\varphi(x)}{x}dx + \int_{\varepsilon_n^-}^{+\infty} \frac{\varphi(x)}{x}dx - \phi(x)\operatorname{Log}(x)|_{x=\varepsilon_n^-}^{\varepsilon_n^+} + \int_{\varepsilon_n^-}^{\varepsilon_n^+} \varphi'(x)\operatorname{Log}(x)dx. \end{aligned}$$

We pass to the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left( \int_{-\infty}^{-\varepsilon_n^-} \frac{\varphi(x)}{x}dx + \int_{\varepsilon_n^-}^{+\infty} \frac{\varphi(x)}{x}dx \right) = \left\langle \text{p.v.} \left(\frac{1}{x}\right), \varphi \right\rangle .$$

$$\begin{aligned} \lim_{n\to\infty} \left( - \phi(x)\operatorname{Log}(x)|_{x=\varepsilon_n^-}^{\varepsilon_n^+} \right) &= \lim_{n\to\infty} \varphi(\varepsilon_n^-)\operatorname{Log}(\varepsilon_n^-) - \varphi(\varepsilon_n^+)\operatorname{Log}(\varepsilon_n^+) \\ &= \lim_{n\to\infty} \varphi(0)\operatorname{Log}\left(\frac{\varepsilon_n^-}{\varepsilon_n^+}\right) + (\varphi(\varepsilon_n^-) - \varphi(0))\operatorname{Log}(\varepsilon_n^-) - (\varphi(\varepsilon_n^+) - \varphi(0))\operatorname{Log}(\varepsilon_n^+) \\ &= -\varphi(0)\operatorname{Log} a, \end{aligned}$$

where in the last equality we used

$$|(\varphi(\varepsilon_n^+) - \varphi(0))\operatorname{Log}(\varepsilon_n^+)| \le \varepsilon_n^+ |\operatorname{Log}(\varepsilon_n^+)| \operatorname{Sup} |\varphi'(x)| \to 0$$

and similarly for $\varepsilon_n^-$. The last term can be estimated as follows:

$$\left| \int_{\varepsilon_n^-}^{\varepsilon_n^+} \varphi'(x)\operatorname{Log}(x)dx \right| \le |\varepsilon_n^+ - \varepsilon_n^-| \operatorname{Sup} |\varphi'| \left( |\operatorname{Log}(\varepsilon_n^+)| + |\operatorname{Log}(\varepsilon_n^-)| \right) \to 0.$$

The conclusion is that $T_n$ converges to the distribution p.v. $\left(\frac{1}{x}\right) - \operatorname{Log}(a)\delta$.

**Exercise 5.** Let $\varphi \in \mathcal{D}(\mathbb{R})$. We compute

$$\langle g'', \varphi \rangle = \int_{\mathbb{R}} g(x)\varphi''(x)\,dx = \int_0^\infty x\varphi''(x)\,dx = -\int_0^\infty \varphi'(x)\,dx = \varphi(0),$$

which shows that $g'' = \delta$ in $\mathcal{D}'(\mathbb{R})$.