Distribution & Interpolation Spaces – Solution sheet 4

Exercise 1. Let $\lambda \in \mathbb{R}$. We have

$$\begin{aligned}\|\lambda\varphi\|_{m,k} &= \max\left\{|D^\alpha(\lambda\varphi)|\,:\,|\alpha|\leq k,\, x\in\Omega\setminus K_m\right\}\\ &= |\lambda|\max\left\{|D^\alpha(\varphi)|\,:\,|\alpha|\leq k,\, x\in\Omega\setminus K_m\right\}\\ &= |\lambda|\|\varphi\|_{m,k}\,.\end{aligned}$$

Moreover, for every $\varphi_1,\varphi_2\in C_c^\infty(\Omega)$

$$\begin{aligned}\|\varphi_1+\varphi_2\|_{m,k} &= \max\left\{|D^\alpha(\varphi_1+\varphi_2)|\,:\,|\alpha|\leq k,\, x\in\Omega\setminus K_m\right\}\\ &\leq \max\left\{|D^\alpha(\varphi_1)|+|D^\alpha(\varphi_2)|\,:\,|\alpha|\leq k,\, x\in\Omega\setminus K_m\right\}\\ &\leq \|\varphi_1\|_{m,k}+\|\varphi_2\|_{m,k}\,.\end{aligned}$$

Exercise 2. Equivalently, we show that

$$\mathcal{D}_{\varphi,\boldsymbol{\epsilon},\boldsymbol{m}} = \mathcal{D}(\Omega)\cap\left\{\psi : \|\psi-\varphi\|_{\boldsymbol{\varepsilon},\boldsymbol{m}} = \sup_{n\in\mathbb{N}}\left(\frac{1}{\epsilon_n}\sup_{|\alpha|\leq m_n}\|D^\alpha\varphi\|_{L^\infty(\Omega\setminus K_n)}\right)<1\right\}$$

is a basis generating a topology on $\mathcal{D}(\Omega)$. We need to show that the sets $D_{\varphi,\boldsymbol{\varepsilon},\boldsymbol{m}}$ cover the whole space $C_c^\infty(\Omega)$ and that for every element in the intersection of two sets of the basis, there exists an another set of the basis containing that element and such that it is itself contained in the intersection.

By choosing $\varepsilon_n =$, $m_n = 0$ for all $n\in\mathbb{N}$ we get that

$$\bigcup_{\varphi\in C_c^\infty(\Omega)}\left\{\psi\in C_c^\infty(\Omega)\,:\;\;\|\varphi-\psi\|_{L^\infty(\Omega)}<1\right\} = C_c^\infty(\Omega).$$

Let now $D_{\varphi,\varepsilon,m}$ and $D_{\tilde{\varphi},\tilde{\varepsilon},\tilde{m}}$ be two elements of the basis with a nonempty intersection (otherwise it is trivial). Let $\gamma\in D_{\varphi,\varepsilon,m}\cap D_{\tilde{\varphi},\tilde{\varepsilon},\tilde{m}}$. Since $\forall n\in\mathbb{N}$ $\|\varphi-\gamma\|_{\boldsymbol{\varepsilon},\boldsymbol{m}}<\varepsilon$ and $\|\tilde{\varphi}-\gamma\|_{\boldsymbol{\varepsilon},\boldsymbol{m}}<1$, then for every $n\in\mathbb{N}$ there exist $\epsilon_n,\tilde{\epsilon}_n>0$ such that

$$\begin{aligned}\|\varphi-\gamma\|_{m_n,n} &:= \sup_{|\alpha|\leq m_n}\|D^\alpha(\varphi-\gamma)\|_{L^\infty(\Omega\setminus K_n)}<\varepsilon_n-\epsilon(n)\\ \|\tilde{\varphi}-\gamma\|_{\tilde{m}_n,n} &= \sup_{|\alpha|\leq \tilde{m}_n}\|D^\alpha(\tilde{\varphi}-\gamma)\|_{L^\infty(\Omega\setminus K_n)}<\tilde{\varepsilon}_n-\tilde{\epsilon}(n),\end{aligned}$$

for all $n\in\mathbb{N}$. We now choose $M_n=\max(m_n,\tilde{m}_n)$ and $E_n=\min(\epsilon_n,\tilde{\epsilon}_n)$. If $\psi\in D_{\gamma,E,M}$ then

$$\begin{aligned}\|\varphi-\psi\|_{m_n,n} &\leq \|\varphi-\gamma\|_{m_n,n}+\|\gamma-\psi\|_{m_n,n}\\ &< \varepsilon_n-\epsilon(n)+\|\gamma-\psi\|_{m_n,n}\\ &< \varepsilon_n\end{aligned}$$

and analogously

$$\|\tilde{\varphi} - \psi\|_{m_n,n} < \tilde{\varepsilon}_n$$

This shows that $D_{\gamma,E,M} \subset D_{\varphi,\varepsilon,m} \cap D_{\tilde{\varphi},\tilde{\varepsilon},\tilde{m}}$, where $E = \{E_n\}_n$ and $M = \{M_n\}_n$

Exercise 3. Suppose that $\exists K \subset \Omega$ such that $\operatorname{spt}\varphi_n \subset K$ and $\varphi_n \to \varphi$ uniformly, together with all of its derivatives. Choose ε and m arbitrarily and let $D_{\varphi,\varepsilon,m}$ be an element of the basis given in the previous exercise. Since $K_m \uparrow \Omega$, then there exists $M \in \mathbb{N}$ such that $(\Omega \setminus K_m) \cap K = \emptyset$ for all $m > M$. We define

$$\overline{m} = \max_{n \le M} m_n,$$
$$\overline{\varepsilon} = \min_{n \le M} \varepsilon_n.$$

Since $\varphi_n \to \varphi$ uniformly (together with all of its derivatives), there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\|\varphi_n - \varphi\|_{C^{\overline{m}}(\Omega)} < \overline{\varepsilon}.$$

This implies that $\forall n > N$, $\varphi_n \in D_{\varphi,\varepsilon,m}$. Note that one should also check that for any other set $D_{\psi,\varepsilon,m}$ containing φ, then the sequence is definitively contained in $D_{\psi,\varepsilon,m}$. However, this is an easy consequence of the fact that in exercise 4 we proved that those sets form a basis for the topology. Indeed we proved the existence of a set $D_{\varphi,\overline{\varepsilon},\overline{m}} \subset D_{\psi,\varepsilon,m} \cap D_{\varphi,\varepsilon,m}$ and from the previous computations we conclude that for some $N > 0$, the sequence $\varphi_n \in D_{\varphi,\overline{\varepsilon},\overline{m}}$, for every $n > N$.

We first prove the existence of the compact set K such that $\operatorname{spt}\varphi_n \subset K$, for all $n \in \mathbb{N}$. If by contradiction there is no K_M containing all the supports of φ_n, then the set

$$\Sigma = \Big\{ j \in \mathbb{N} \,:\, \exists n_j \text{ and } n(j) \,:\, \|\varphi_{n_j} - \varphi\|_{j,n(j)} := \sup_{|\alpha| \le j} \|D^\alpha(\varphi - \varphi_{n_j})\|_{L^\infty(\Omega \setminus K_{n(j)})} > 0 \Big\}$$

contains countably many integers. We can thus define

$$\varepsilon_j = 1, \quad \mathit{if}\; j \in \mathbb{N} \setminus \Sigma$$
$$\varepsilon_j = \frac{\|\varphi_{n_j} - \varphi\|_{j,n(j)}}{2}, \quad \mathit{if}\; j \in \Sigma.$$

Then we deduce that $\varphi_{n_j} \notin \mathcal{D}_{\varphi,\varepsilon,m}$ for all $j \in \Sigma$, where $m_j = j$ for all $j \in \mathbb{N}$, which means that there are infinitely many elements of the sequence that are not contained in $\mathcal{D}_{\varphi,\varepsilon,m}$. This is of course in contradiction with the assumption that the sequence was topologically converging to φ. Fix a multi-index α such that $|\alpha| = j$, a real number $\epsilon > 0$ and choose

$$m_n = j \qquad \varepsilon_n = \frac{\epsilon}{M},$$

$\forall n \in \mathbb{N}$. By hypothesis there exists $\overline{n} = \overline{n}(\epsilon) \in \mathbb{N}$ such that

$$\|\varphi_n - \varphi\|_{\boldsymbol{\varepsilon},\boldsymbol{m}} < \frac{\epsilon}{M} \quad \forall n > \overline{n},$$

from which we conclude that

$$\|D^\alpha \varphi_n - D^\alpha \varphi\|_{L^\infty(\Omega)} \le \sum_{m=0}^{M-1} \|\varphi_n - \varphi\|_{\varepsilon,m} < \frac{\epsilon}{M} M = \epsilon\,.$$

where the first inequality follows from the definition of $\|\cdot\|_{\boldsymbol{\varepsilon},\boldsymbol{m}}$ with ε_0 and m_0.

Exercise 4. Suppose that Ω is a bounded domain. Thus $\partial\Omega \neq \emptyset$. Since T does not have compact support in Ω, there exists a sequence of points x_k accumulating on $\partial\Omega$ and a sequence $\epsilon_k > 0$ such that $B_{\epsilon_k}(x_k) \subset \Omega$ and

$$\langle T, \psi_k \rangle > 0, \quad \textit{for some } \psi_k \in C_c^\infty(B_{\epsilon_k}(x_k)).$$

We define $\varphi_k = k \frac{\psi_k}{\langle T, \psi_k \rangle}$. Since $\operatorname{spt} \varphi_k \subset \operatorname{spt} \psi_k \subset B_{\epsilon_k}(x_k)$, we have that $\varphi_k \to 0$ in $\mathcal{E}(\Omega)$, but

$$\langle T, \psi_k \rangle = k \to \infty.$$

In the case Ω unbounded the points x_k could also diverge to infinity, but the proof still runs in very same way.

Exercise 5. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We compute

$$\begin{aligned}
\left\langle \frac{d^2}{dt^2} \left[(H(t) - H(t-2)) (t^2 - t - 2) \right], \varphi \right\rangle &= \int_{\mathbb{R}} (H(t) - H(t-2)) (t^2 - t - 2) \varphi''(t) \, dt \\
&= \int_0^\infty (t^2 - t - 2) \varphi''(t) \, dt - \int_2^\infty (t^2 - t - 2) \varphi''(t) \, dt \\
&= 2\varphi'(0) - \int_0^\infty (2t - 1) \varphi'(t) \, dt + \int_2^\infty (t^2 - t - 2) \varphi'(t) \, dt \\
&= 2\varphi'(0) - \varphi(0) - 3\varphi(2) + 2 \int_{\mathbb{R}} (H(t) - H(t-2)) \varphi(t) \, dt,
\end{aligned}$$

from which we deduce that

$$\frac{d^2}{dt^2} \left[(H(t) - H(t-2)) (t^2 - t - 2) \right] = -2\delta_0' - \delta_0 - 3\delta_2 + 2(H(t) - H(t-2)).$$

We denote $g(t) = \frac{1 - \cos(2\pi t)}{t}$ and consequently $g'(t) = \frac{2\pi t \sin(2\pi t) - 1 + \cos(2\pi t)}{t}$. Note that $g(k) = 0$ for any $k \in \mathbb{Z}$ and let us compute g'. For every $t \neq 0$ it holds

$$g'(t) = \frac{2\pi t \sin(2\pi t) - (1 - \cos(2\pi t))}{t^2}.$$

So for every $k \in \mathbb{Z} \backslash \{0\}$ we have $g'(k) = 0$ and, recalling $\lim_{t \to 0} \frac{1 - \cos t}{t^2} = \frac{1}{2}$, we obtain $g'(0) = 2\pi^2$. Therefore for all $\varphi \in \mathcal{D}(\mathbb{R})$ we compute

$$\left\langle g(t) \sum_{k \in \mathbb{Z}} \delta_k', \varphi \right\rangle = \sum_{k \in \mathbb{Z}} \langle \delta_k', g\varphi \rangle = - \sum_{k \in \mathbb{Z}} (g'(k)\varphi(k) + g(k)\varphi'(k)) = -g'(0)\varphi(0) = -2\pi^2 \varphi(0).$$

Thus $g(t) \sum_{k \in \mathbb{Z}} \delta_k' = -2\pi^2 \delta$.