Distribution & Interpolation Spaces – Solution sheet 3

Exercise 1. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^n)$. Then

$$\begin{aligned}\delta_0(\lambda_1\varphi_1 + \lambda_2\varphi_2) &= (\lambda_1\varphi_1 + \lambda_2\varphi_2)(0)\\ &= \lambda_1\varphi_1(0) + \lambda_2\varphi_2(0)\\ &= \lambda_1\delta_0(\varphi_1) + \lambda_2\delta_0(\varphi_2),\end{aligned}$$

which shows that δ_0 is linear. Moreover

$$|\delta_0(\varphi)| = |\varphi(0)| \leq \|\varphi\|_{L^\infty(\mathbb{R}^n)}, \tag{1}$$

which implies the continuity over $C_c(\mathbb{R}^n)$.

By using (1) and the fact that $C_c(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, by the Hahn-Banach theorem we can extend δ_0 to a functional in $(L^\infty(\mathbb{R}^n))'$.

Suppose now that there exists $f \in L^1(\mathbb{R}^n)$ such that

$$\langle \delta_0, g\rangle_{L^\infty(\mathbb{R}^n)' \leftrightarrow L^\infty(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f g\, dx, \quad \forall g \in L^\infty(\mathbb{R}^n).$$

For every $g \in C_c(\mathbb{R}^n \setminus \{0\})$ then

$$\int_{\mathbb{R}^n} f g\, dx = \langle \delta_0, g\rangle_{L^\infty(\mathbb{R}^n)' \leftrightarrow L^\infty(\mathbb{R}^n)} = \delta_0(g) = g(0) = 0,$$

and by using exercise 2 from the exercise sheet 2 we deduce that $f = 0$ for almost every $x \in \mathbb{R}^n$, which in turn implies that $f = 0$ for almost every $x \in \mathbb{R}^n$. Choose now $g \in C_c(\mathbb{R}^n)$ such that $g(0) \neq 0$. We have

$$0 = \int_{\mathbb{R}^n} f g\, dx = \langle \delta_0, g\rangle_{L^\infty(\mathbb{R}^n)' \leftrightarrow L^\infty(\mathbb{R}^n)} = \delta_0(g) = g(0) \neq 0,$$

which is a contradiction.

Exercise 2. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C_c(\mathbb{R})$. Then

$$\begin{aligned}\delta_0'(\lambda_1\varphi_1 + \lambda_2\varphi_2) &= -(\lambda_1\varphi_1 + \lambda_2\varphi_2)'(0)\\ &= -\lambda_1\varphi_1'(0) - \lambda_2\varphi_2'(0)\\ &= \lambda_1\delta_0'(\varphi_1) + \lambda_2\delta_0'(\varphi_2),\end{aligned}$$

which shows that δ_0' is linear. Moreover

$$\left|\langle \delta_0', \varphi\rangle_{\mathcal{D}(\mathbb{R})' \leftrightarrow \mathcal{D}(\mathbb{R})}\right| = |\varphi'(0)| \leq \|\varphi\|_{C^1(\mathbb{R})},$$

which implies that δ_0' is continuous on $\mathcal{D}(\mathbb{R})$.

Moreover, if there exists a measure μ such that

$$\langle \delta_0', \varphi \rangle_{\mathcal{D}(\mathbb{R})' \leftrightarrow \mathcal{D}(\mathbb{R})} = \int_{\mathbb{R}} \varphi \, d\mu,$$

then by choosing $\varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$

$$\int_{\mathbb{R}} \varphi \, d\mu = \langle \delta_0', \varphi \rangle_{\mathcal{D}(\mathbb{R})' \leftrightarrow \mathcal{D}(\mathbb{R})} = -\varphi'(0) = 0,$$

which would imply that $\mu(A) = 0$ for every $A \subset (\mathbb{R} \setminus \{0\})$. Thus μ is concentrated on $\{0\}$, which means

$$\varphi(0)\mu(\{0\}) = \int_{\mathbb{R}} \varphi \, d\mu = -\varphi'(0). \tag{2}$$

By choosing $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi \equiv 1$ on $B_1(0)$, we see that the relation (2) implies $\mu(\{0\}) = 0$. Thus $\mu(A) = 0$ for every $A \subset \mathbb{R}$, from which we conclude that such a measure can not exists, since in general $\varphi'(0) \neq 0$.

Exercise 3. Let α be any multi-index and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Since $T_k \rightharpoonup T$ we have

$$\langle D^\alpha T_k, \varphi \rangle = (-1)^{|\alpha|} \langle T_k, D^\alpha \varphi \rangle \to (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle = \langle D^\alpha T, \varphi \rangle.$$

Thus $D^\alpha T_k \rightharpoonup D^\alpha T$.

Exercise 4. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have that

$$T_k(\varphi) = \delta_{\frac{1}{k}}(\varphi) = \varphi(\frac{1}{k}) \to \varphi(0) = \delta_0(\varphi),$$

which shows that $T_k \rightharpoonup \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Moreover

$$\begin{aligned} S_k(\varphi) = k\left(T_k(\varphi) - T_{2k}(\varphi)\right) &= \frac{\varphi(\frac{1}{k}) - \varphi(\frac{1}{2k})}{\frac{1}{k}} \\ &= \frac{\varphi(\frac{1}{k}) - \varphi(0)}{\frac{1}{k}} - \frac{1}{2}\frac{\varphi(\frac{1}{2k}) - \varphi(0)}{\frac{1}{2k}} \to \varphi'(0) - \frac{\varphi'(0)}{2} = -\frac{\delta_0'(\varphi)}{2}, \end{aligned}$$

which shows that $S_k \rightharpoonup \frac{\delta_0'}{2}$.

Exercise 5. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We compute

$$\delta_0'' e^t(\varphi) = \delta_0''(e^t\varphi) = \delta_0((e^t\varphi)'') = \delta_0(e^t(\varphi + 2\varphi' + \varphi'')) = \varphi(0) + 2\varphi'(0) + \varphi''(0),$$

from which we deduce $\delta_0'' e^t = \delta_0 - 2\delta_0' + \delta_0''$.

We now compute $f_{a,b}'$. For any $\varphi \in \mathcal{D}(\mathbb{R})$ we have that

$$\begin{aligned} \langle f_{a,b}', \varphi \rangle = -\langle f_{a,b}, \varphi' \rangle &= -\lim_{\epsilon \to 0} \int_{|x| > \epsilon} f_{a,b}(x)\varphi'(x)\, dx \\ &= -\lim_{\epsilon \to 0} \left[\int_\epsilon^\infty \mathrm{Log}(ax)\varphi'(x)\, dx + \int_{-\infty}^{-\epsilon} \mathrm{Log}(-bx)\varphi'(x)\, dx \right]. \end{aligned}$$

Integrating by parts we get

$$\int_{\epsilon}^{\infty} \mathrm{Log}(ax)\varphi'(x)\,dx = -\mathrm{Log}(a\epsilon)\varphi(\epsilon) - \int_{\epsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx,$$
$$\int_{-\infty}^{-\epsilon} \mathrm{Log}(-bx)\varphi'(x)\,dx = \mathrm{Log}(b\epsilon)\varphi(-\epsilon) - \int_{-\infty}^{-\epsilon} \frac{1}{x}\varphi(x)\,dx,$$

from which we deduce

$$\langle f'_{a,b}, \varphi\rangle = \lim_{\epsilon\to 0}\left[\mathrm{Log}(a\epsilon)\varphi(\epsilon) - \mathrm{Log}(b\epsilon)\varphi(-\epsilon)\right] + \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{1}{x}\varphi(x)\,dx.$$

Moreover

$$\begin{aligned}\mathrm{Log}(a\epsilon)\varphi(\epsilon) - \mathrm{Log}(b\epsilon)\varphi(-\epsilon) &= \mathrm{Log}(a)\varphi(\epsilon) - \mathrm{Log}(b)\varphi(-\epsilon) + \mathrm{Log}(\epsilon)(\varphi(\epsilon) - \varphi(-\epsilon))\\ &\to \left(\mathrm{Log}(a) - \mathrm{Log}(b)\right)\varphi(0) = \mathrm{Log}\left(\frac{a}{b}\right)\varphi(0).\end{aligned}$$

On the other hand the term $\lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{1}{x}\varphi(x)\,dx$ corresponds to a distribution in $\mathcal{D}'(\mathbb{R})$ usually called *principal value* defined as

$$\langle p.v.\left(\frac{1}{x}\right), \varphi\rangle_{\mathcal{D}'\leftrightarrow\mathcal{D}} = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{1}{x}\varphi(x)\,dx = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}dx.$$

Thus we conclude that

$$f'_{a,b} = \mathrm{Log}\left(\frac{a}{b}\right)\delta_0 + p.v.\left(\frac{1}{x}\right).$$