EPFL - Autumn semester 2022-2023	J. Scherer
Homotopy theory	**Sheet 9**
Mathematics	14th November 2022

Exercise 1. A contractibility condition.

1. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_n \hookrightarrow X_{n+1}$ is nullhomotopic.

2. Show that the infinite sphere $S^\infty = \{x \in \mathbb{R}^\infty \mid \sum_i x_i^2 = 1\}$ is contractible. Here $\mathbb{R}^\infty$ is the union $\bigcup_n \mathbb{R}^n$.

3. More generally, show that $\Sigma^\infty X$ is contractible for any CW complex X. Here $\Sigma^\infty X$ denotes the union $\bigcup_n \Sigma^n X$.

Proof.

1. If $\varphi : S^n \to X$ is a pointed map, because S^n is compact, φ factors through $X_k \hookrightarrow X$ for some k. But since $X_k \hookrightarrow X_{k+1}$ is nullhomotopic, the composite $S^n \to X_k \hookrightarrow X$ is nullhomotopic, hence $\pi_n(X) \cong 0$ for all n. By Whitehead's theorem, we conclude that X is contractible.

2. The infinite dimensional sphere S^∞ is the union of the sequence $S^0 \subset S^1 \subset \cdots$, where each $S^n \subset S^{n+1}$ is nullhomotopic. We conclude by part 1. that S^∞ is contractible.

3. Similarly, Σ^∞ is the union of the sequence $X \subset \Sigma X \subset \Sigma^2 X \subset \cdots$ where each $\Sigma^n X \subset \Sigma^{n+1} X$ is a subcomplex inclusion. It remains to show that $Y \hookrightarrow \Sigma Y$ is nullhomotopic for any space Y. But such a nullhomotopy is given by $h_t(y) = \overline{(y, \frac{t+1}{2})}$.

□

◯**Exercise 2. Weak equivalence as an equivalence relation.**

Recall that a weak equivalence is a map $f : X \to Y$ that induces isomorphisms $f_* : \pi_n(X, x) \cong \pi_n(Y, f(x))$ for any $x \in X$. Let $\sim$ be the smallest equivalence relation on spaces such that if there is a weak equivalence $X \to Y$, then $X \sim Y$.

1. Show that $X \sim Y$ if and only if there is a finite zig-zag of weak equivalences $X \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_k \to Y$.

2. Show that $X \sim Y$ if and only if X and Y have a common CW approximation.

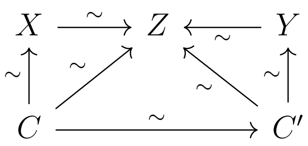
3. If X and Y are path-connected, is it true that $X \sim Y$ if and only if $\pi_n(X, x) \cong \pi_n(Y, y)$ for all n and any basepoints $x \in X$, $y \in Y$?

4. If X and Y are path-connected, is it true that $X \sim Y$ if and only if $H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z})$ for all n? You can use $\mathbb{C}P^2$ and $S^2 \vee S^4$ to try this out.

Proof.

1. Symmetry of the relation impose weak equivalence in opposite direction $Y \leftarrow X$, and transitivity imposes to have zigzags.

2. If X and Y have a common CW approximation, then there is a CW complex C with weak equivalences $X \overset{\sim}{\leftarrow} C \overset{\sim}{\rightarrow} Y$, so that $X \sim Y$. Conversely, if $X \sim Y$, we can assume without loss of generality that there is a zig-zag of the form $X \overset{\sim}{\leftarrow} Z \overset{\sim}{\rightarrow} Y$. Chose CW approximations $C \overset{\sim}{\rightarrow} X, C' \overset{\sim}{\rightarrow} Y$ for X and Y. We get two CW approximations $C \overset{\sim}{\rightarrow} X \overset{\sim}{\rightarrow} Z \overset{\sim}{\leftarrow} Y \overset{\sim}{\leftarrow} C'$ for Z. By uniqueness of CW approximations, there is a weak equivalence $C \overset{\sim}{\rightarrow} C'$ making the diagram



commute, which gives a common CW approximation for X and Y.

3. False. We have seen that $S^2 \times \mathbb{R}P^\infty$ and $\mathbb{R}P^2$ have the same homotopy groups but are not weakly equivalent.

4. The two pairs have trivial homology groups, except in degree $0, 2, 4$ where it is just $\mathbb{Z}$. However they don't have the same third homotopy groups. We know that the pair $(S^2 \times S^4, S^2 \vee S^4)$ is 4-connected (sheet 5), hence $\pi_3(S^2 \vee S^4) \cong \pi_3(S^2 \times S^4) \cong \pi_3(S^2) \cong \mathbb{Z})$. We know that the complex projective plane has exactly one cell in dimension $0, 2, 4$ (hence the homology groups). By the cellular approximation theorem, we know that any pointed map $f : S^3 \to \mathbb{C}P^2$ is based homotopic to a map $S^3 \overset{\alpha}{\to} S^2 \subseteq \mathbb{C}P^2$. Hence $[f]$ is the image of $1 \in \pi_3(S^3) = \mathbb{Z}$ by the composition $\pi_3(S^3) \overset{\alpha}{\to} \pi_3(S^2) \to \pi_3(\mathbb{C}P^2)$. However we claim that the second map is trivial. This is because the third cell of $\mathbb{C}P^2$ is attached along the Hopf map $S^3 \to S^2$, so that $S^3 \overset{\eta}{\to} S^2 \subseteq \mathbb{C}P^2$ is nullhomotopic. But $\eta : S^3 \to S^2$ is a generator of $\pi_3(S^2) \cong \mathbb{Z}$, hence $\pi_3(S^2) \to \pi_3(\mathbb{C}P^2)$ is trivial. We conclude that $[f] = 0$ in $\pi_3(\mathbb{C}P^2)$ and thus that the latter group is trivial.

□

◯**Exercise 3. Uniqueness of highly connected covers.**
Let X be a path connected and pointed space. Recall that a pointed map $X\langle n\rangle \to X$ is called an n-connected cover if it induces an isomorphism on π_k for $k > n$ and $\pi_k X\langle n\rangle = 0$ for $k \leq n$. Show that two n-connected covers are weakly equivalent.

Hint. You can use our favorite model for $X\langle n\rangle$ constructed from a single point by attaching cells of dimension $\geq n+1$ and compare it to any other n-connected cover in the spirit of the comparison of CW-approximations. Conclude by checking directly that the comparison map is a weak equivalence and by the description of the equivalence relation from Exercise 2.

Proof. Exactly as in the mentioned proof. □

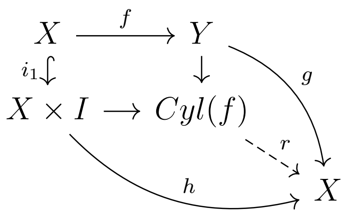
Exercise 4. Relating retracts and homotopy retracts.

1. Show that a map $f : X \to Y$ has a left homotopy inverse if and only if $X \hookrightarrow Cyl(f)$ has a retraction.

2. Show that f is a homotopy equivalence if and only if $X \hookrightarrow Cyl(f)$ is a deformation retract.

3. Can you find a map $f : X \to Y$ such that $X \hookrightarrow Cyl(f)$ has a retraction but is not a deformation retract?

Proof. 1. If $h : X \times I \to X$ is a homotopy $g \circ f \simeq id_X$, define a map $r : Cyl(f) \to X$ by the universal property



of the pushout defining $Cyl(f)$. Denoting $i : X \hookrightarrow Cyl(f)$ the inclusion, $r \circ i$ is the composite $X \xrightarrow{i_0} X \times I \to Cyl(f) \xrightarrow{r} X$, which equals the composite $X \xrightarrow{i_0} X \times I \xrightarrow{h} X$ which is just $h(-, 0) = id_X$. Conversely, suppose $i : X \hookrightarrow Cyl(f)$ has a retraction r. Restricting r to $Y \to Cyl(f)$ gives a map $g : Y \to X$, and the composite $X \times I \to Cyl(f) \xrightarrow{r} X$ provides a homotopy $g \circ f \simeq id_X$.

2.

3. From the first parts we only have to find a map having a left homotopy inverse, but no homotopy inverse. Choose for example $S^1 \hookrightarrow S^1 \vee S^1$. This is not a homotopy equivalence.

□

◊ indicates the weekly assignments.