EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 8**
Mathematics	8 April 2025

Exercise 1. Different homotopy types that have the same homotopy groups.
Define $\mathbb{R}P^\infty$ as the union $\bigcup_{n\geq 0}\mathbb{R}P^n$ where the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ is induced by the inclusion $S^n \hookrightarrow S^{n+1}$ given by $(x_0,\cdots x_n)\mapsto(x_0,\cdots x_n,0)$.

Show that $\mathbb{R}P^2$ and $S^2\times\mathbb{R}P^\infty$ have the same homotopy groups, but are not homotopy equivalent.

Proof. Note that since S^2 is the universal cover of $\mathbb{R}P^2$, they have the same homotopy groups in degree $k\geq 2$. By the cellular approximation theorem, every map $S^k\to\mathbb{R}P^\infty$ factors, up to (based) homotopy, through the k-skeleton $\mathbb{R}P^k\subseteq\mathbb{R}P^{k+1}\subseteq\mathbb{R}P^\infty$. For $k\geq 2$, $\pi_k(\mathbb{R}P^{k+1})\cong\pi_k(S^{k+1})=0$, which shows that $\pi_k(\mathbb{R}P^\infty)=0$ for all $k\geq 2$. By the same argument, we have a surjective map $\mathbb{Z}/2\cong\pi_1(\mathbb{R}P^2)\to\pi_1(\mathbb{R}P^\infty)$. Let $f:S^1\to\mathbb{R}P^2\subseteq\mathbb{R}P^\infty$ be a generator. If $H:S^1\times I\to\mathbb{R}P^\infty$ was nullhomotopy of f, the homotopy would factor through $\mathbb{R}P^2$, a contradiction. Hence $\pi_1(\mathbb{R}P^\infty)\cong\mathbb{Z}/2$.
By the Künneth formula, or by computing cellular homology, they have a different second homology group.

□

Exercise 2. Wedge of cofibrations is a cofibration.
If $i:A\to X$ and $j:B\to Y$ are cofibrations of pointed spaces, show that $i\vee j:A\vee B\to X\vee Y$ is a cofibration.

Proof. Suppose that we have a homotopy $H:(A\vee B)\times I\to Z$, and a map $f:X\vee Y\to Z$ extending such that $f\circ(i\vee j)=H_0$. Restricting the homotopy to $H_A=H|_{A\times I}$ and $H_B:=H|_{B\times I}$, and extending them to $H_X:X\times I\to Z$ and $H_Y:Y\times I\to Z$ we obtain a homotopy $(H_X\vee H_Y):(X\times I)\vee(Y\times I)\to Z$ (since $H_X(a_0,0)=H_A(a_0,0)=H(a_0,0)=H(b_0,0)=H_Y(b_0,0)$). But since the homotopies are pointed, meaning that $H_X(a_0,t)=H_A(a_0,t)=H(a_0,t)=H_Y(b_0,t)$ for all $t\in I$, the homotopy factors through $(X\vee Y)\times I$ as desired. □

◊Exercise 3. Making the identity laws of an H-space strict.

1. If X and Y are finite CW complexes, show that $X\vee Y\to X\times Y$ is a cofibration.
   Remark: *The conclusion still holds when X and Y are only assumed to be well-pointed.*

2. Let (X,x) be an H-space and a pointed CW complex. Show that the multiplication $\mu: X\times X\to X$ is homotopic to a map m for which the base point is a strict identity.

3. Apply this strictification to ΩX (no explicit formula is expected).

4. There is another more explicit way to do this. Define the Moore loop space $\Omega_M X$ as the subspace of $map_*([0,\infty),X)\times[0,\infty[$ consisting of those pairs (α,t) where α is a path starting at x_0 and t is such that $\alpha(s)=x_0$ for all $s\geq t$, i.e. α becomes constant at x_0 after some time t. Show that $\Omega X\simeq\Omega_M X$, and define a product on $\Omega_M(X)$ such that the identity law is strict.

Remark. The composition law you defined is also strictly associative.

Proof. 1. $X \vee Y \to X \times Y$ is the inclusion of a subcomplex, hence is a cofibration by the lecture. For well pointed spaces, the two points inclusion $i : 1 \to X$ and $j : 1 \to Y$ are cofibrations, hence their pushout-product $i \square j : X \vee Y \to X \times Y$ is a cofibration.

2. By the strictification lemma to the cofibration $X \vee X \to X \times X$ and the collapse map $X \vee X \to X$.

3. If X is a finite CW-complex, then ΩX is a loop spaace by a theorem of Milnor [Mil59]. If X is only well pointed, then ΩX is also well pointed [Str11, §5.6]. The space ΩX is an H-group, in particular an H-space. Hence we know there exists a strictly unital composition $\Omega X \times \Omega X \to \Omega X$.

4. Define $\varphi : \Omega X \to \Omega_M X$ by $\alpha \mapsto (\alpha, 1)$ where we extended α by the constant map on $[1, \infty[$. Define $\psi : \Omega_M X \to \Omega X$ by $(\alpha, t) \mapsto \alpha(t \cdot -)$. Then ψ is a strict left inverse of φ, while $\varphi \cdot \psi \simeq 1_{\Omega_M X}$ by the homotopy $H : \Omega_M X \times I \to \Omega_M X$ defined by

$$((\alpha, t), s) \mapsto \Big(\alpha\big((t(1-s)+s)\cdot -\big), ts + (1-s)\Big)$$

Define a concatenation $\Omega_M X \times \Omega_M X \to \Omega_M X$ by setting $(\alpha, t) * (\beta, t') = (\alpha * \beta, t + t')$ where

$$\alpha * \beta(s) = \begin{cases} \alpha(s) & s \leq t \\ \beta(s-t) & s \geq t. \end{cases}$$

This operation is clearly strictly unital and strictly associative, where the unit is $(c_{x_0}, 0) \in \Omega_M X$.

□

◊**Exercise 4. Cofibrations and the extension problem.** Let $i : A \hookrightarrow X$ be a cofibration, and $f \simeq g : A \to Z$ two homotopic maps.

1. Show that f extends to a map $X \to Z$ if and only if g does.

2. Let $\varphi : Z \xrightarrow{\sim} Z'$ be a homotopy equivalence. Show that f extends to a map $X \to Z$ if and only if $\varphi \circ f$ extends to a map $X \to Z'$.

3. Let $f_0, f_1 : X \to Z$ be maps that agree on A, ie. $f_0|_A = f_1|_A$. Show that if $\varphi \circ f_0 \simeq \varphi \circ f_1$ rel. A, then $f_0 \simeq f_1$ rel. A.

Proof. 1. It's a direct application of the HEP.

2. Suppose that $f : A \to Z$ extends to $\tilde{f} : X \to Z$, then $\varphi \cdot \tilde{f}$ is an extension of $\varphi \cdot f$. Now suppose that $\widetilde{\varphi \cdot f}$ is an extension of $\varphi \cdot f$. Then $\psi \cdot \widetilde{\varphi \cdot f}$ is an extension of $\psi \cdot \varphi \cdot f$, where ψ is a homotopy inverse of φ. Since $\psi \cdot \varphi \cdot f \simeq f$, there exists an extension of f by the first point.

3. Consider the two cofibrations $\{0,1\} \hookrightarrow I$ and $A \hookrightarrow X$. Their pushout product $A \times I \cup X \times \{0,1\} \hookrightarrow X \times I$ is a cofibration by the lectures. We apply the second point to this cofibration. Consider $H : A \times I \cup X \times \{0,1\} \to Z$ to be constant homotopy at $f_0|_A = f_1|_A$ on $A \times I$, and $f_0 \sqcup f_2$ on $X \times \{0,1\}$. By assumption $\varphi \circ H$ extends to a map from $X \times I$, hence the second point tells us that H extends to a map $\tilde{H} : X \times I \to Z$. This new homotopy is a homotopy $f_0 \simeq_{rel\ A} f_1$ as desired.

□

Exercise 5. Cofiber sequence induced by a composable pair.
Let $f : X \to Y$ and $g : Y \to Z$ be pointed maps. Write $C(f)$ for the mapping cone of a map f.

1. Define canonical maps $\alpha : C(f) \to C(g \circ f)$ and $\beta : C(g \circ f) \to C(g)$.

2. Show that $C(f) \xrightarrow{\alpha} C(g \circ f) \xrightarrow{\beta} C(g)$ is a cofiber sequence, ie. that β is the mapping cone of the map α.

Proof. We give a categorical proof. One could also write down formulas.

1. Consider the following diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
{\scriptstyle i_0}\downarrow & & \downarrow & & \downarrow \\
CX & \longrightarrow & C(f) & \xrightarrow{\alpha} & C(f \circ g) \\
 & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta} \\
 & & CY & \longrightarrow & C(g)
\end{array}$$

By definition, the space $C(f)$ is the pushout of the left square, while $C(g \circ f)$ is the pushout of the horizontal rectangle. Hence the top right square is also a pushout and α is the canonical map. Now consider the universal map $j : C(f) \to CY$ induced by the maps $Cf : CX \to CY$ and $Y \to CY$, and construct the pushout of the bottom square. It follows that the vertical rectangle is a pushout, hence we obtain $C(g)$ by definition, where β is the canonical map.

2. To come!

□

◊ indicates the weekly assignments.