EPFL - Autumn semester 2022-2023 — J. Scherer
Homotopy theory — **Sheet 7**
MATHEMATICS — 1st November 2022

Exercise 1. A map homotopic to the identity.
Let X be a pointed space. Define a map $f : \Sigma X \to \Sigma X$ by the formula $\overline{(x,t)} \mapsto \overline{(x, \min(2t,1))}$. Show that $f \simeq \mathrm{id}_{\Sigma X}$ is homotopic to the identity.

Proof. A homotopy $f \simeq id_{\Sigma X}$ is given by $f_s : \overline{(x,t)} \mapsto \overline{(x, \min(t(1+s), 1)}$. □

◊Exercise 2. Pushout squares preserve quotients.
Recall that an embedding is an injective map $j : A \to X$ which induces a homeomorphism $A \cong j(A)$ onto its image. Suppose the left square in the following diagram is a pushout with j an embedding.

$$\begin{array}{ccccc} A & \xrightarrow{j} & X & \xrightarrow{p} & X/A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \overline{F}} \\ B & \xrightarrow{J} & Y & \xrightarrow{q} & Y/B \end{array}$$

1. Show that J is also an embedding.
2. Define the induced map $\overline{F} : X/A \to B/Y$ on the quotients.
3. Show that $\overline{F}$ is a homeomorphism.

Remark. If we work with CW-complexes, the cofibers X/A and Y/B are called homotopy cofibers of j and J respectively, and the pushout is a homotopy pushout. We will see that the converse is also true: the left square is a homotopy pushout if and only if the homotopy cofibers are equivalent! The same is true for homotopy pullback and homotopy fibers.

Proof.
1. The pushout of an injective map is injective. Recall that an injective map $\iota : C \to D$ is an embedding if for all open $U \subseteq C$ there exists an open $W \subseteq D$ such that $\iota^{-1}(W) = U$. Since j is an embedding, we treat it as a subspace inclusion $A \subseteq X$. Now let $U \subseteq B$ be open. Since j is an embedding, there exists an open $V \subseteq X$ such that $V \cap A = f^{-1}(U)$. Let $W := q(U \sqcup V) \subseteq Y$ which is open. Then $J(U) = J(B) \cap W$, i.e. $U = J^{-1}(W)$. Hence J is an embedding.
2. Consider the composition $X \to Y \to Y/B$. Since the left square commute, A is sent to the base point of Y/B and so the maps factors through $\overline{F} : X/A \to Y/B$.
3. We give a categorical proof. Consider the following diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & Y/B \end{array}$$

The two squares are pushouts. But pasting two pushout squares yields a pushout square, so that the rectangle is also a pushout. But the usual model for the pushout of the rectangle is X/A. Hence the unique map $X/A \to Y/B$ given by the universal property of the pushout is an isomorphism in Top, i.e. a homeomorphism.

□

◊Exercise 3. The h-coaction on the mapping cone.

Let $f : X \to Y$ be a map of pointed spaces. Write $C(f)$ for the mapping cone of f defined as $CX \cup_f Y$ where $CX = X \wedge I$ where we take 0 to be the basepoint of I. Define a map $\mu : C(f) \to \Sigma X \vee C(f)$ by

$$\overline{(x,t)} \mapsto \begin{cases} (\,\overline{(x,2t)}\,, *) & \text{if } 0 \le t \le \frac{1}{2} \\ (*, \overline{(x,2t-1)}\,) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

and $y \mapsto y$.

1. Show that for any pointed space Z, the map μ induces a left action of the group $[\Sigma X, Z]_*$ on the pointed set $[C(f), Z]_*$. Here $[-,-]_* = \pi_0 Map_*(-,-)$ denotes homotopy classes of pointed maps.

2. When f is the inclusion $X \hookrightarrow CX$, show that this action can be identified with left multiplication on the group $[\Sigma X, Z]_*$.

Proof. Notice that the map μ collapses $X \times \{\frac{1}{2}\} \subseteq C(f)$. The lower part of $C(f)$ in the quotient is isomorphic to ΣX (parametrized by $[0, \frac{1}{2}]$), while the upper part is $C(f)$ (parametrized by $[\frac{1}{2}, 1]$).

1. Since μ is a pointed map, it induces a map $[\Sigma X, Z]_* \times [C(f), Z]_* \cong [\Sigma X \vee C(f), Z]_* \xrightarrow{\mu} [C(f), Z]_*$. Explicitly, the action of $[\alpha : \Sigma X \to Z]$ on $[f : C(f) \to Z]$ is given by $[C(f) \xrightarrow{\mu} \Sigma X \vee C(f) \xrightarrow{(\alpha,f)} Z]$. If $[\alpha] = [cst_{z_0} : \Sigma X \to Z]$ the identity element, the product is the class of the map $C(f) \to Z$ which is constant on the lower part ($\le \frac{1}{2}$) of the cone $C(f)$, and is exactly f on the upper part. This is based homotopic to f by

$$H : C(f) \times I \to Z$$

$$(\overline{(x,t)}, s) \mapsto \begin{cases} z_0 & 0 \le t \le \dfrac{s}{2} \\ f\overline{(x, \dfrac{2t-s}{2-s})} & \dfrac{s}{2} \le t \le 1 \end{cases}$$

Now if we have $[\alpha], [\beta] \in [\Sigma, Z]_*$, the product $(\alpha \cdot \beta) \cdot f$ is the class of $C(f) \to Z$ which is α on the $[0, \frac{1}{4}]$ part of the cone, β on $[\frac{1}{4}, \frac{1}{2}]$, and g on $[\frac{1}{2}, 1]$. On the other hand $\alpha \cdot (\beta \cdot g)$ is α on $[0, \frac{1}{2}]$, β on $[\frac{1}{2}, \frac{3}{4}]$ and g on $[\frac{3}{4}, 1]$. Those two decomposition of the interval are homotopic (the one we use to prove the associativity of path concatenation), hence we can use it to provide the desired homotopy.

2. Notice that in this case $C(f) = \Sigma X$ (parametrized by the interval $[-1, 1]$), and μ is the usual comultiplication $\Sigma X \to \Sigma X \vee \Sigma X$ which collapses the suspension in the middle (at $0 \in [-1, 1]$).

$\square$

$\Diamond$Exercise 4. The fiber sequence of a map.
(Only questions (1), (2), (3) are part of the assignment)
Let $f : X \to Y$ be a map of pointed spaces and consider the sequence of iterated mapping fibers (homotopy fibers)

$$\cdots \longrightarrow F(f_2) \xrightarrow{f_3} F(f_1) \xrightarrow{f_2} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y.$$

Here $f_1 : F(f) \to X$ is the mapping (homotopy) fiber of f, $f_2 : F(f_1) \to F(f)$ is the mapping (homotopy) fiber of f_1, and so on.

1. $\Diamond$ Describe the elements of $F(f_1)$ and its topology, together with the map $f_2 : F(f_1) \to F(f)$.
2. $\Diamond$ Define maps $\phi_Y : \Omega Y \to F(f_1)$ and $\psi_Y : F(f_1) \to \Omega Y$ that form a pointed homotopy equivalence. Write $j := f_2 \cdot \phi_Y : \Omega Y \to F(f)$.
3. $\Diamond$ Deduce that $F(f_2) \simeq \Omega X$ are homotopy equivalent and show that the following diagram is pointed homotopy commutative

$$\begin{array}{ccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y \\ {\scriptstyle \phi_X}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle \psi_Y} \\ F(f_2) & \xrightarrow[f_3]{} & F(f_1) \end{array}$$

   where $-\Omega f = \iota \cdot \Omega f$ and $\iota : \Omega X \to \Omega X$ is the inversion map $\omega \mapsto \overline{\omega}$.
4. Deduce that the sequence

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{j} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$$

   is h-exact.
5. Deduce that the following sequence is h-exact

$$\cdots \longrightarrow \Omega^2 F(f) \xrightarrow{\Omega^2 f_1} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega j} \Omega F(f) \xrightarrow{-\Omega f_1} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{j} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y.$$

We know make some observations that will help you have new insights on this exercise, once we have developed the theory of fibrations and homotopy pullbacks.

6. Show that the fiber $f_1^{-1}(x_0)$ is homeomorphic to ΩY, and that the inclusion $\Omega Y \subset F(f)$ is the map j (hence j is an embedding). This tells us that the strict fiber of f_1 is also the homotopy fiber.
7. Likewise, show that the fiber $f_2^{-1}((x_0, c_{y_0}))$ is homeomorphic to ΩX and that the inclusion $\Omega Y \subset F(f_1)$ is $f_3 \cdot \phi_X$. However notice that ΩX is definitely not the fiber of $j : \Omega Y \to F(f)$ (it is its *homotopy* fiber!).

Proof. 1. The iterated construction tells us that elements of $F(f_1)$ are triples $(x,\omega,\gamma) \in X \times F(X) \times F(Y)$ where $\gamma(1) = f(x)$ and $\omega(1) = x$. It is topologized as a subspace of $X \times PY \times PX$. Considering the following pasting of pullbacks

$$\begin{array}{ccccc} F(f_1) & \xrightarrow[f_2]{} & F(f) & \longrightarrow & F(Y) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_1} & \lrcorner & \downarrow{\scriptstyle ev_1} \\ F(X) & \xrightarrow[ev_1]{} & X & \xrightarrow[f]{} & Y \end{array} \tag{1}$$

we observe that it is homeomorphic to $F(X) \times_Y F(Y) = \{(\omega,\gamma) \in F(X) \times F(Y) | \; \gamma(1) = f(\omega(1))\}$. The map f_2 is then given by $(\omega,\gamma) \mapsto (\omega(1),\gamma)$.

2. Define ϕ and ψ by the formulas $\phi(\gamma) = (c_{x_0},\gamma)$ and $\psi(\omega,\gamma) = \gamma * \overline{f(\omega)}$ where $\overline{f(\omega)}$ is the path $f(\omega)$ in reverse direction. Define pointed homotopies $H : \phi \circ \psi \simeq id_{F(f_1)}$ and $K : \psi \circ \phi \simeq id_{\Omega Y}$ by $H_s(\omega,\gamma) = (\omega(s \cdot -), (\gamma * \overline{f(\omega)})((1 - \frac{s}{2}) \cdot -))$ and $K_s(\gamma) = \gamma(\min((1+s) \cdot -, 1))$.

3. Replacing $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ by $F(f_1) \xrightarrow{f_2} F(f) \xrightarrow{f_1} X$ in the previous point yields the first claim. Notice that $F(f_2) \cong FX \times_X F(F(f))$ with $f_3(\omega,\lambda) \mapsto (\omega, \lambda_2(1))$, where $\lambda = (\lambda_1,\lambda_2)$ since it is a path in a (fibered) product.

   The bottom composition is

$$\begin{aligned} \psi_Y \cdot f_3 \cdot \phi_X(\omega) &= \psi_Y \cdot f_3(\omega, c_{(x_0,c_{y_0})}) = \psi_Y(\omega, c_{y_0}) \\ &= c_{y_0} * \overline{f(\omega)}, \end{aligned}$$

   while $-\Omega f(\omega) = \overline{f(\omega)}$. A homotopy is given as in the second point.

4. Consider the following diagram

$$\begin{array}{ccccccccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y & & & & & & \\ {\scriptstyle \phi_X}\downarrow & & {\scriptstyle \psi_Y}\uparrow\downarrow{\scriptstyle \phi_Y} & \searrow^{j} & & & & & \\ F(f_2) & \xrightarrow[f_3]{} & F(f_1) & \xrightarrow[f_2]{} & F(f) & \xrightarrow[f_1]{} & X & \xrightarrow[f]{} & Y \end{array}$$

   We know that the bottom sequence is h-exact (iterate Lemma 2.4 of chapter 4, which I let you prove). The vertical maps are pointed equivalences (by 2.), the square is pointed commutative (by 3.), and the triangle commutes. Since we are considering pointed homotopy equivalence classes of maps $[-,Z]_*$, the desired sequence is h-exact as well.

5. By induction, noticing that $-\Omega(-\Omega f) = \Omega^2 f$.

6. Consider the right pullback square of (1). As in the proof of exercise 2.3, the fiber of f_1 is homeomorphic to the fiber of $ev_1 : F(Y) \to Y$, which is precisely ΩY. With these identifications, the inclusion $\Omega Y \subset F(f)$ is given by $\gamma \mapsto (x_0,\gamma)$, which is precisely j.

7. Similarly the fiber of f_2 is ΩX. However the fiber of j is a point since j is injective.

□

Exercise 5*. Relating fiber and cofiber sequences.
Let $f : X \to Y$ be a map of pointed spaces. Define a map $\zeta : F(f) \to \Omega C(f)$ by the formula

$$\zeta(x,\gamma)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \overline{(x, 2t-1)} & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Define also $\xi : \Sigma F(f) \to C(f)$ as the adjoint of ζ. That is, $\xi(x,\gamma,t) = \zeta(x,\gamma)(t)$. Recall that the adjunction $\Sigma \dashv \Omega$ provides maps $\eta_X : X \to \Omega\Sigma X$ and $\varepsilon_X : \Sigma\Omega X \to X$ natural in X.

In the following diagram, the top row is obtained from the fiber sequence of f by application of the functor Σ, and the bottom row is obtained by applying Ω to the cofiber sequence of f.

$$\begin{array}{ccccccccccc}
 & & \Sigma\Omega F(f) & \xrightarrow{\Sigma\Omega p} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma\alpha} & \Sigma F(f) & \xrightarrow{\Sigma p} & \Sigma X \\
 & & \downarrow{\scriptstyle \varepsilon_{F(f)}} & & \downarrow{\scriptstyle \varepsilon_X} & & \downarrow{\scriptstyle \varepsilon_Y} & & \downarrow{\scriptstyle \xi} & & \| \\
\Omega Y & \xrightarrow{\alpha} & F(f) & \xrightarrow{p} & X & \xrightarrow{f} & Y & \xrightarrow{i} & C(f) & \xrightarrow{\pi} & \Sigma X \\
\| & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_{C(f)}} & & \\
\Omega Y & \xrightarrow[\Omega i]{} & \Omega C(f) & \xrightarrow[\Omega\pi]{} & \Omega\Sigma X & \xrightarrow[\Omega\Sigma f]{} & \Omega\Sigma Y & \xrightarrow[\Omega\Sigma i]{} & \Omega\Sigma C(f) & &
\end{array}$$

Show that the diagram is homotopy commutative. Which of the squares commute strictly?
Hint: there are only two explicit homotopies to write.

Proof. Let's label the squares (1)-(8) from left to right starting by the first line. The squares (1-2-7-8) are naturality squares for ε and η so commute strictly. The square (3) is adjoint to (5), while (4) is adjoint to (6), so one only needs to check (5) and (6). For (5), the composite $\zeta \circ \alpha$ sends $\gamma \in \Omega Y$ to $\gamma * c_{(x_0, c_{y_0})}$ while Ωi is $\gamma \mapsto \gamma$. A homotopy $H : \zeta \circ \alpha \simeq \Omega i$ is given by $H_s(\gamma)(t) = \gamma(\min(2t, 1))$. For (6) we have that $\eta_X \circ p(x,\gamma) = \eta_X(x)$ is the loop $t \mapsto (x,t) \in \Sigma X$, while $\Omega\pi \circ \zeta(x,\gamma)$ is the loop

$$t \mapsto \begin{cases} (x,0) & t \leq \frac{1}{2} \\ (x, 2t-1) & \frac{1}{2} \leq t. \end{cases}$$

A homotopy $L : \eta_X(x) \simeq \Omega\pi \circ \zeta(x,\gamma)$ is given by $L_s(x,\gamma) = \left(t \mapsto \begin{cases} (x,0) & t \leq \frac{1}{2}s \\ (x, (2t-1)s + t(1-s)) & \frac{1}{2}s \leq t \end{cases}\right).$ □

◊ indicates the weekly assignments.