EPFL - Autumn semester 2022-2023 — J. Scherer

Homotopy theory — **Sheet 6**

MATHEMATICS — 24th October 2022

Exercise 1. Examples of CW approximations.

1. Find a CW approximation to the quasi-circle (defined in sheet 3).
2. Find a CW approximation to the space $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\} \subset \mathbb{R}$.

Proof. 1. We have seen that $\pi_n(Q) = 0$ for all n in a previous problem sheet. Hence $* \to Q$ is a CW approximation.

2. The map $\mathbb{N} \to \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ defined by $n \mapsto \frac{1}{n}$ and $0 \mapsto 0$ is a CW approximation since it induces a bijection on π_0 and higher homotopy groups of both spaces are trivial.

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◊Exercise 2. A space with prescribed homotopy groups.

1. Comparing $S^n \vee S^n$ with $S^n \times S^n$ compute $\pi_n(S^n \vee S^n)$ (the case $n = 1$ has to be treated separately).
2. Compute $\pi_n(\bigvee_{i=1}^n S^n)$ for any finite wedge of spheres (and an arbitrary wedge if you want!).
3. Let $M(\mathbb{Z}/p^k, n)$ be the $(n+1)$-dimensional Moore space $S^n \cup_{p^k} e^{n+1}$, where the top cell is attached via the degree p^k map on S^n. Show that $\pi_n M(\mathbb{Z}/p, n) \cong \mathbb{Z}/p$ by first showing it must be a quotient of $\pi_n S^n$ and then using cellular homology to identify the order of the generator. Can you do it for $M(\mathbb{Z}/p^k, n)$?
4. For any finitely generated abelian group A, construct a connected space X with $\pi_n X \cong A$.

Proof. 1. By exercise 3 of sheet 5, we know that $\pi_n(S^n \vee S^n) \cong \mathbb{Z} \times \mathbb{Z}$ for all $n \geq 2$. Seifert van-Kampen shows that $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$.

2. Using a similar argument, we find that $\pi_n(\bigvee_{i=1}^n S^n) \cong \mathbb{Z}^n$.

3. We prove the general case directly: let $M = M(\mathbb{Z}/p^k, n)$ and $i : S^n \hookrightarrow M$ the inclusion of the n-skeleton. By exercise 3 of sheet 5, the pair (M, S^n) is n-connected, which shows that $i_* : \pi_n S^n \to \pi_n M$ is surjective. Hence $\pi_n M \cong \mathbb{Z}/\ker i_*$ is a quotient of $\mathbb{Z}$. Notice first that since

$$\begin{array}{ccc} S^n & \xrightarrow{p^k} & S^n \\ \big\downarrow & & \big\downarrow i \\ D^{n+1} & \longrightarrow & M \end{array}$$

commutes (it is a pushout), $i_*([p^k]) = [i \circ p^k] = 0 \in \pi_n(M)$. It follows that $p^k\mathbb{Z} \subseteq ker(i_*)$. We deduce that $ker(i_*) = p^l\mathbb{Z}$ for some $l \leq p$. Suppose for a contradiction that $l < k$, so that the composition $f : S^n \xrightarrow{p^l} S^n \xrightarrow{i} X$ is (based) nullhomotopic. But on cellular homology, we can easily observe (using exercise 6 of sheet 5) that $H_n^{\text{cell}}(f)(1) = p^l \in \mathbb{Z}/p^k\mathbb{Z} \cong H_n^{\text{cell}}(M)$ which is non-zero. One sees this by computing $H_n^{\text{cell}}(M)$ first, and using the formula for functoriality. It follows that f can't be nullhomotopic, otherwise we would have $H_n(f) = 0$. This shows that $k = l$ and $\pi_n(M) \cong \mathbb{Z}/p^k\mathbb{Z}$.

4. Suppose that $A \cong \bigoplus_I \mathbb{Z} \oplus \mathbb{Z}/p^{k_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{k_n}\mathbb{Z}$. We define
$$X = \bigvee_I S^n \vee M(\mathbb{Z}/p^{k_1}\mathbb{Z}, n) \vee \cdots \vee M(\mathbb{Z}/p^{k_n}\mathbb{Z}, n)$$
which has the desired property.

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◊Exercise 3. Equivalent definitions of n-connectedness.
We say that a map $f : X \to Y$ is an n-equivalence if it induces isomorphisms $f_* : \pi_k(X,x) \cong \pi_k(Y, f(x))$ for $k < n$ and a surjection $\pi_n(X,x) \twoheadrightarrow \pi_n(Y, f(x))$ for $k = n$, for any basepoint $x \in X$. Show that the following properties of a space X are equivalent :

1. $\pi_0 Map_*(K, (X,x)) = 0$ for any pointed CW complex K of dimension $\leq n$ and any $x \in X$.
2. $X \to *$ is an $(n+1)$-equivalence.
3. $x : * \to X$ is an n-equivalence for any $x \in X$.
4. $\pi_k(X,x) = 0$ for all $k \leq n$ and any $x \in X$.

Hint: For (4) ⇒ (1), show by induction on $m \leq n$ that $K^m \hookrightarrow K \to X$ is nullhomotopic.

Proof. Conditions (2), (3), (4) are easily seen to be equivalent since $0 \to G$ being surjective amounts to G being trivial. For (1) ⇒ (4), just apply (1) where $K = S^k$ for $k \leq n$. Now assuming (4), let $f : K \to X$ a based map (we assume WLG that the base point of K is in K^0). We prove inductively on $m \leq n$ that $f^m : K^m \hookrightarrow K \xrightarrow{f} X$ is based nullhomotopic. This will show the result since $K^n = K$. For $m = 0$ the 0-skeleton K^0 is just a discrete set. Since X is path connected, a choice of paths between points in the image of f^0 to $x \in X$ gives a based nullhomotopy. Now suppose that $f^m := f|_{K^m}$ is based nullomotopic for $m < n$. Since (K^{m+1}, K^m) has the homotopy extension property (exercise 1.2 of sheet 5), and $f^{m+1}|_{K^m} = f^m$, we can extend the nullhomotopy of f^m to K^{m+1}. We obtain that f^{m+1} is based homotopic to a map $K^{m+1} \to X$ which sends K^m to a point, i.e. factors through $K^{m+1}/K^m \cong \bigvee S^{m+1}$. But $[\bigvee S^{m+1}, X]_* \cong \prod [S^{m+1}, X]_* \cong \prod \pi_{m+1}(X,x) = 0$, which implies that $K^{m+1}/K^m \to X$ is based nullhomotopic. Precomposing with the quotient map $K^{m+1} \to K^{m+1}/K^m$ we find that f^m is based nullhomotopic as well. □

◊Exercise 4. The Whitehead tower of a space.
In this exercise all spaces are pointed.

1. Given a space X and $n \geq 0$, build an n-connected CW complex $X\langle n\rangle$ together with a map $f_n : X\langle n\rangle \to X$ that induces isomorphisms $\pi_k(X\langle n\rangle) \cong \pi_k(X)$ for all $k > n$.
   Hint: Reproduce the proof of CW approximation seen in class.
2. Modify the construction of $X\langle n\rangle$ so that for each n, there are maps $X\langle n+1\rangle \to X\langle n\rangle$ making the triangle $\begin{array}{ccc} X\langle n+1\rangle & \longrightarrow & X\langle n\rangle \\ & {\scriptstyle f_{n+1}}\searrow \quad \swarrow{\scriptstyle f_n} & \\ & X & \end{array}$ strictly commutative.

   Hint: construct $X\langle n+1\rangle$ from $X\langle n\rangle$. The resulting diagram $\cdots \to X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X\langle 0\rangle \to X$ is called a Whitehead tower for X.

Proof. 1. We apply the construction of the CW-approximation, but starting at level $n+1$ in order to obtain a space with trivial homotopy groups in degree $\leq n$. Explicitly, we start with a wedge $\bigvee_I S^{n+1}$ of $(n+1)$-spheres, where I is a set of generators of $\pi_{n+1}(X)$, instead of a wedge of circles. This comes with map to X corresponding to the generators. Then attach $(n+2)$-cells to kill the kernel, and continue as in the lecture notes.

2. We proceed by induction. Having $X\langle n\rangle \to X$ in hands, we apply the construction of the first point to $X\langle n\rangle$ itself to obtain $(X\langle n\rangle)\langle n+1\rangle \to X\langle n\rangle$. Note that $(X\langle n\rangle)\langle n+1\rangle = X\langle n+1\rangle$ by construction, but this time it comes with a map $X\langle n+1\rangle \to X\langle n\rangle$ as desired.

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◊Exercise 5. Some relations between a CW complex and its n-skeleton.
Recall that if X and Y are CW complexes, the product $X \times Y$ has a canonical CW structure.

1. Let X be an n-connected space. Show that there exists a CW approximation $K \to X$ such that K has trivial n-skeleton, ie. $K^{(n)} = *$.

2. Let (X, x) be a pointed CW complex. Show that the inclusion $X^{(n)} \hookrightarrow X$ of the n-skeleton is an n-equivalence, ie. induces isomorphisms $\pi_k(X^{(n)}, x) \cong \pi_k(X, x)$ for $k < n$ and a surjection $\pi_n(X^{(n)}, x) \twoheadrightarrow \pi_n(X, x)$.

3. If $X \simeq Y$ are homotopy equivalent CW complexes without cells of dimension $n+1$, show that $X^{(n)} \simeq Y^{(n)}$. Disprove the statement for arbitrary CW complexes.

Proof. 1. The construction done in class is such that $X^{(n)} = *$.

2. The pair $(X, X^{(n)})$ is n-connected by an exercise of sheet 5, so we conclude by the long exact sequence.

3. Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverses. By cellular approximation, we can suppose that both maps are cellular. Let $H : X \times I \to X$ be a homotopy $gf \simeq 1_X$. Since H is already cellular on $X \times \{0,1\}$, H is homotopic to a cellular map $H' : X \times I \to X$ relative to $X \times \{0,1\}$. Hence H' is a cellular homotopy $gf \simeq 1_X$. Consider the restriction $H'|_{X^{(n)} \times I} : X^{(n)} \times I \to Y$. The domain is a CW-complex of dimension $n+1$, so it lands in $Y^{(n+1)} = Y^{(n)}$ by assumption. It follows that $H'|_{X^{(n)} \times I}$ is a homotopy $f^{(n)}g^{(n)} \simeq 1_{X^{(n)}}$. Similarly $g^{(n)}f^{(n)} \simeq 1_{Y^{(n)}}$.

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◊ indicates the weekly assignments.