EPFL - Autumn semester 2022-2023	J. Scherer
Homotopy theory	**Sheet 5**
MATHEMATICS	17th October 2022

◊Exercise 1. Cellular approximation for pairs.

1. Show that the inclusion $S^{n-1} \times I \cup D^n \times 0 \subset D^n \times I$ is a strong deformation retract.

2. Prove that the inclusion $A \hookrightarrow A \cup_f e^n = X$ verifies the HEP (Homotopy Extension Property): Given a map $f \colon X \to Z$ and a homotopy $G \colon A \times I \to Z$ starting at $f|_A$, there exists a homotopy $F \colon X \times I \to Z$ starting at f and extending G.

3. Show that every map $f : (X, A) \to (Y, B)$ of CW pairs is homotopic through maps $(X, A) \to (Y, B)$ to a cellular map.

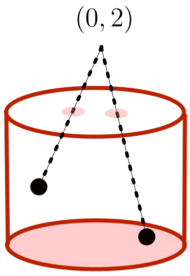
Proof. 1. Write $\iota : S^{n-1} \times I \cup D^n \times \{0\} \subseteq D^n \times I$ for the inclusion and define $r : D^n \times I \longrightarrow S^{n-1} \times I \cup D^n \times \{0\}$ by

$$r(x, t) = (\chi(x, t)x, 2 + \chi(x, t)(t - 2))$$

where $\chi : S^{n-1} \times I \cup D^n \times \{0\} \longrightarrow \mathbb{R}$ is defined by

$$\chi(x, t) = \begin{cases} \frac{2}{2-t} & \text{if } 2\|x\| \leq 2 - t \\ \frac{1}{\|x\|} & \text{if } 2\|x\| \geq 2 - t \end{cases}.$$

This corresponds to the projection of the cylinder on $S^{n-1} \times I \cup D^n \times \{0\}$ from the point $(0, 2) \in \mathbb{R}^n \times \mathbb{R}$, as depicted below.

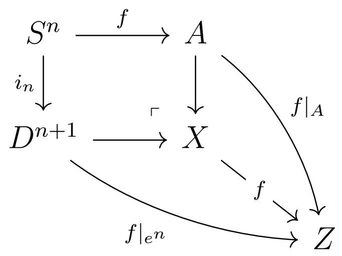


We need to prove that $r \circ \iota = Id$ and $\iota \circ r \simeq Id$. The first equality is a simple verification. For the second one, it is clear from the picture, but a formula is given by

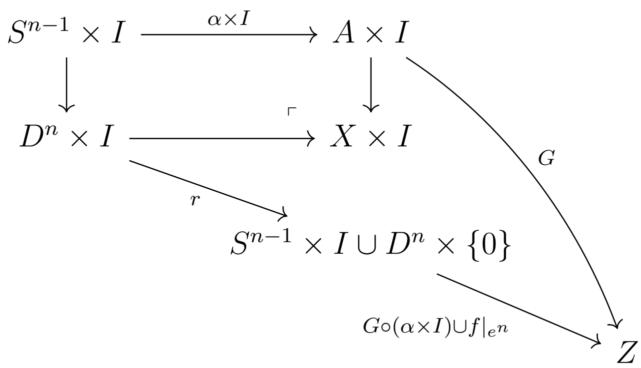
$$\begin{array}{rcl} H : \quad (D^n \times I) \times I & \longrightarrow & D^n \times I \\ (x, t, s) & \longmapsto & \Big([(1 - s) + s\chi(x, t)]x, 2 + [(1 - s) + s\chi(x, t)](t - 2)\Big) \end{array}$$

A simple verification shows that H is the desired homotopy, and that it fixes $S^{n-1} \times I \cup D^n \times \{0\}$.

2. The space X together with the map f can be given by the following pushout diagram:



Since $(- \times I) \dashv Map(I, -)$ we know that $- \times I$ preserves pushout diagrams. It follows that to give a homotopy $F : X \times I \to Z$, we only need to provide a homotopy $G : A \times I \to Z$ and a map $D^n \times I \to Z$ which agree on $S^n \times I$. We do it as follow:



The bottom map $G \circ (\alpha \times I) \cup f|_{e^n}$ is well defined since both maps agree on $S^{n-1} \times \{0\}$. It is immediate that the diagram commute since r is a retraction of ι by the first point. Hence there exists $F : X \times I \to Z$ extending F, starting at $F|_{X \times \{0\}} = F|_{A \times \{0\}} \cup F|_{D^n \times \{0\}} = f|_A \cup f|_{e^n} = f$ as desired.

3. First use the cellular approximation theorem to find a homotopy $H : A \times I \to B$ between $f_{|A}^{|B}$ and a cellular map $g : A \to B$. Then since (X, A) has the homotopy extension property, we can extend the composite $A \times I \xrightarrow{H} B \hookrightarrow Y$ to a homotopy $\tilde{H} : X \times I \to Y$ with $\tilde{H}(-, 0) = f$ and $\tilde{H}(-, 1)_{|A}^{|B} = g$. Applying the relative cellular approximation theorem, we can find a homotopy relative to A $K : \tilde{H}(-, 1) \simeq f'$ rel. A, where $f' : X \to Y$ is cellular. It follows that f' is now cellular as pairs $(X, A) \to (Y, B)$, and composite $\tilde{H} \cdot K$ gives a homotopy of pairs $f \simeq f'$.

□

Exercise 2. Connectivity of some pairs. A pair (X, A) is n-connected if $\pi_k(X, A, a) = 0$ for all $k \leq n$ and any basepoint $a \in A$. For $k = 0$, we defined $\pi_0(X, A) = \pi_0(X)/i_*(\pi_0(A))$ where $i_* : \pi_0(A) \to \pi_0(X)$ is induced by the inclusion. The condition says here that A intersects every path component of X.

1. What are the connectivities of the pairs (S^n, S^k),$(\mathbb{R}P^n, \mathbb{R}P^k)$ and $(\mathbb{C}P^n, \mathbb{C}P^k)$ for $n > k$?

2. If X is a CW complex with $X^{(n)} = *$, what can you say about the connexity of ΣX and of the pair $(\Sigma X, X)$?

Proof. 1. Use the standard cell decomposition of $S^n, \mathbb{R}P^n$ and $\mathbb{C}P^n$ and identify $S^k, \mathbb{R}P^k$ and $\mathbb{C}P^k$ respectively as the k, k and $2k$-skeletons. The three pairs are k, k and $2k$-connected respectively. To see this you can use the following exercise 3.3, or do the computation directly. The pairs are not $(k+1)$, $(k+1)$ and $(2k+1)$-connected since it would imply by the long exact sequence that $\pi_k(S^k) = \pi_k(\mathbb{R}P^k) = \pi_{2k}(\mathbb{C}P^k) = 0$.

2. Since ΣX si a quotient of $X \times I$, the n-cells of X are in bijective correspondance with the $(n+1)$-cells of ΣX. Hence if X only has cells of dimension $> n$, ΣX only has cells of dimension $> n+1$, so is $(n+1)$-connected. Since ΣX is obtained as a pushout of $CX \leftarrow X \rightarrow CX$, we find that ΣX is obtained from X by attaching $(n+1)$-cells, so the pair $(\Sigma X, X)$ is n-connected. One can also see this via the long exact sequence for the pair.

□

◊Exercise 3. Connectivity of some more pairs.

1. Show that if X and Y are CW complexes with $X^{(m)} = *$ and $Y^{(n)} = *$, then the pair $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected, as is the smash product $X \wedge Y$.
2. Prove that the inclusion $S^n \vee S^n \hookrightarrow S^n \times S^n$ induces an isomorphism on π_k for all $k < 2n-1$ (and a surjection if $k = 2n-1$).
3. For X a CW complex, show that the pair $(X, X^{(n)})$ is n-connected. Here $X^{(n)}$ denotes the n-skeleton of X.

Proof. 1. In the standard cell structure for $X \times Y$ the $(k+\ell)$-cells are indexed by $e_X^k \times e_Y^\ell$. The cells of the form $* \times e_Y^\ell$ and $e_X^k \times *$ for $k \geq m$ and $\ell \geq n$ form a cell decomposition of $X \vee Y$ while the remaining cells are of dimension $> m+n+1$. Hence $(X \times Y)^{(n+m+1)} \subseteq X \vee Y$ is the $(m+n+1)$-skeleton of $X \times Y$. Now the same technique as in the a)-proof of the third point allows us to conclude directly. For the same reason as above, $X \wedge Y = X \times Y / X \vee Y$ has only cells of dimension $> n+m+1$ so is $(n+m+1)$-connected.

2. By the first point, $(S^n \times S^n, S^n \vee S^n)$ is $2n+1$-connected. The long exact sequence for this pair concludes.

3. We give two different proofs.

   (a) We apply the relative cellular approximation of maps of pairs to some $f : (D^k, S^{k-1}) \rightarrow (X, X^{(n)})$ for $k \leq n$ which represents an element in $\pi_k(X, X^{(n)}, x_0)$. Since any connected component contains at least a 0-cell, it is harmless to suppose that the base point $x_0 \in X$ is a 0-cell. It follows that f is cellular on $\{x_0\} \subseteq X^{(n)}$, hence is homotopic, through a homotopy of pairs and relative to $\{x_0\}$ (a pointed homotopy), to a cellular map $f' : (D^k, S^{k-1}) \rightarrow (X, X^{(n)})$. Hence f' factors through $(X^{(n)}, X^{(n)})$ and hence is nullhomotopic. This proves that $\pi_k(X, X^{(n)}, x_0) = 0$ and therefore the pair $(X, X^{(n)})$ is n-connected.

   (b) A different strategy is to prove that the inclusion $X^{(n)} \rightarrow X$ induces isomorphism on π_k for all $0 \leq k < n$ and a surjection for $k = n$. This would imply the result by the long exact sequence of a pair. The surjectivity on π_k follows from the fact that a map $S^k \rightarrow X$ is homotopic, relative to the base point, to a cellular map $S^k \rightarrow X^{(n)} \subseteq X$. For injectivity suppose that a map $f : S^k \rightarrow X^{(n)}$ is nullhomotopic when seen as a map

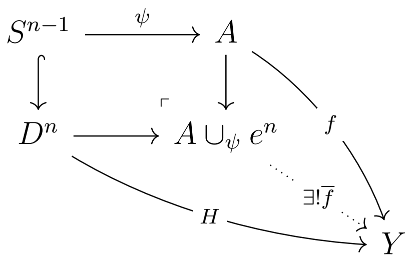
$S^k \to X$, i.e. there exists $H : S^k \times I \to X$ a pointed nullhomotopy. Since H is cellular on $S^k \times \{0,1\} \cup \{s_0\} \times I$, and $S^k \times I$ is a CW-complex of dimension $k+1$, H is homotopic, relative to $S^k \times \{0,1\} \cup \{s_0\} \times I$ to a cellular map $S^k \times I \to X^{(n)}$. But this gives a pointed nullhomotopy of f. This shows injectivity.

□

◊Exercise 4. An extension criterion.

1. Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.
2. Show that a CW complex retracts onto any contractible subcomplex: given a CW pair (X, A) where A is contractible, show that there exist $r : X \to A$ with $r \circ \iota = id_A$, where $\iota : A \hookrightarrow X$ is the inclusion.

Proof. 1. If $X \setminus A$ has an n-cell with attaching map $\psi : S^{n-1} \to A$, then f extends to $A \cup_\psi e^n$ if and only if $f \circ \psi$ is nullhomotopic, which is the case if $\pi_n(Y) = 0$. We can see this using the universal property of the pushout



An extension $\overline{f}$ exists if and only if $f \circ \psi$ factors through D^n, i.e. if and only if $f \circ \psi$ is null-homotopic. In general, extend f inductively on the cells of $X \setminus A$ starting with the cells of lowest dimensions.

2. Since A is contractible, the identity $A \to A$ extends by question 1) to a map $r : X \to A$. We have $r \circ i = id_A$ since r is an extention of id_A.

□

◊Exercise 5. The degree of a map $S^n \to S^n$.

The goal of this execrcise is to show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity. Thus the degree of such a map determines its homotopy class.

1. Reduce to the case where there is a point $y \in S^n$ such that $f^{-1}(y) = \{x_1, \cdots, x_k\}$ and f is an invertible linear map in the neighbourhood of each x_i.
   Hint: Use the lemma seen in class on PL approximation of a map $I^n \to Y$ on a polyhedron $K \subset I^n$.
2. For f as in (1), consider the composition $g \circ f$ where $g : S^n \to S^n$ collapses the complement of a small ball around y to the basepoint. Use this to reduce to the case where $k = 1$ in (1).
3. Conclude using the fact that $\mathrm{GL}_n(\mathbb{R})$ has only two path components (the proof of this fact is not required).

Proof. See an attached solution on moodle. □

Exercise 6. Functoriality of cellular homology.

1. Recall the definition of cellular homology $H_*^{\text{cell}}(X)$ for a CW complex X.
2. Using cellular approximation, show that a map $f : X \to Y$ between CW complexes induces well defined group morphisms $f_* : H_*^{\text{cell}}(X) \to H_*^{\text{cell}}(Y)$.
3. Show that for any pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of composable maps, we have $g_* \circ f_* = (g \circ f)_*$.

Proof.

1. If X is a CW complex, the cellular chain complex $C_*^{\text{cell}}(X)$ of X is given in degree n by the free abelian group $\mathbb{Z}[\{e_\alpha^n\}_\alpha]$ on the n-cells of X. The differential is given on an $(n+1)$-cell e_α^{n+1} by $de_\alpha^{n+1} = \sum_\beta d_{\alpha\beta} e_\beta^n$ where $d_{\alpha\beta}$ is the degree of the map
$$S_\alpha^n \xrightarrow{\psi_\alpha} X^{(n)} \to X^{(n)}/X^{(n-1)} \simeq \bigvee_{\beta'} S_{\beta'}^n \to S_\beta^n$$
where ψ_α is the attaching map for the cell e_β^{n+1} and the last map is the one collapsing every $S_{\beta'}^n$ for $\beta' \neq \beta$. Then $H_*^{\text{cell}}(X) = H(C_*^{\text{cell}}(X))$ is the homology of this chain complex.

2. We use cellular approximation to homotope f to a cellular map $g : X \to Y$. Define the map $f'_* : C_*^{\text{cell}}(X) \to C_*^{\text{cell}}(Y)$ as follows : since f' is cellular it induces a map $\overline{f'}^{(n)} : X^{(n)}/X^{(n-1)} \to Y^{(n)}/Y^{(n-1)}$. Now the map g_n between cellular n-chains is defined by $f'_n(e_\alpha^n) = \sum_\beta \delta_{\alpha\beta}\varepsilon_\beta^n$ where ε_β^n is a labelling of the n-cells of Y and $\delta_{\alpha\beta}$ is the degree of the map
$$S_\alpha^n \hookrightarrow \bigvee_\alpha S_\alpha^n \simeq X^{(n)}/X^{(n-1)} \xrightarrow{\overline{f'}^{(n)}} Y^{(n)}/Y^{(n-1)} \simeq \bigvee_\beta S_\beta^n \to S_\beta^n.$$
This defines a map of complexes and hence descends to a map $f'_* : H_*^{\text{cell}}(X) \to H_*^{\text{cell}}(Y)$.

3. Using cellular approximation, f and g are homotopic to cellular maps f' and g' respectively. For an n-cell in X, we have that
$$(g_* \circ f_*)(e_\alpha^n) = g_*(\sum_\beta \delta_{\alpha,\beta} e_\beta^n) = \sum_\beta \delta_{\alpha,\beta} g_*(e_\beta^n) = \sum_\beta \sum_\gamma \delta_{\alpha,\beta}\delta_{\beta,\gamma} e_\gamma^n$$
where $\{e_\gamma^n\}$ is a labeling of the n-cells in Z. Now notice that $g' \circ f'$ is a cellular approximation to $g \circ f$, so that we might use this map to define $(g \circ f)_*$. Let us write
$$(g \circ f)_*(e_\alpha^n) = \sum_\gamma \delta_{\alpha,\gamma} e_\gamma^n,$$
where the coefficients $\delta_{\alpha,\gamma}$ are the degree of the map
$$S_\alpha^n \to \bigvee_\alpha S_\alpha^n \cong X^{(n)}/X^{(n-1)} \to Y^{(n)}/Y^{(n-1)} \to Z^{(n)}/Z^{(n-1)} \cong \bigvee_\gamma S_\gamma^n \to S_\gamma^n \tag{1}$$
We need to show that
$$\sum_{\beta,\gamma} \delta_{\alpha,\beta}\delta_{\beta,\gamma} = \sum_\gamma \delta_{\alpha,\gamma}. \tag{2}$$

Applying $H_n(-)$ to the sequence (1) we obtain

$$\mathbb{Z} \to \bigoplus_{\alpha} \mathbb{Z} \xrightarrow{f'} \bigoplus_{\beta} \mathbb{Z} \xrightarrow{g'} \bigoplus_{\gamma} \mathbb{Z} \to \mathbb{Z}$$

where $f'(e_\alpha) = \sum_\beta \delta_{\alpha,\beta} e_\beta$ and $g'(e_\beta) = \sum_\gamma \delta_{\beta,\gamma} e_\gamma$. It follows that through this composition, 1 is sent to $\sum_\beta \delta_{\alpha,\beta}\delta_{\beta,\gamma}$, which by definition is the degree $\delta_{\alpha,\gamma}$ of the composition (1). The equation (2) now follows directly, which concludes the proof.

□

◊ indicates the weekly assignments.