EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 4**
Mathematics	11 March 2025

Exercice 1. Some examples of relative homotopy groups.

1. Compute the relative homotopy groups of the pair $(S^1 \times S^1, S^1 \vee S^1)$.
2. What can you say about the relative groups of the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$?

Proof. 1. The universal covers of both $S^1 \times S^1$ and $S^1 \vee S^1$ are contractible ($\mathbb{R}^2$ and the Cayley graph of the free group on two generators respectively), so we only have to identify the homomorphism $i_* \colon \pi_1(S^1 \vee S^1) \to \pi_1(S^1 \times S^1)$ induced by the inclusion of the wedge into the product. This is the abelianization $F(x, y) \to \mathbb{Z}^2$.

The long exact sequence in homotopy allows us to conclude that all homotopy groups (or sets) $\pi_n(S^1 \times S^1, S^1 \vee S^1)$ are trivial (zero or a singleton) except for $\pi_2(S^1 \times S^1, S^1 \vee S^1)$. The latter is the kernel of i_*, i.e. a free group on infinitely many generators.

2. The long exact sequence in homotopy tells us that $\pi_n(\mathbb{R}P^2, \mathbb{R}P^1) \cong \pi_n(\mathbb{R}P^2)$ for all $n \geq 3$. For $n = 2$ the long exact sequence takes the form of

$$1 \to \mathbb{Z} \to \pi_2(\mathbb{R}P^2, \mathbb{R}P^1) \to \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathbb{R}P^2, \mathbb{R}P^1) \to 1$$

This tells us that $\pi_1(\mathbb{R}P^2, \mathbb{R}P^1) = 1$ while $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1)$ is a split extension of $\mathbb{Z}$ with itself, i.e. that $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1) \cong \mathbb{Z} \rtimes \mathbb{Z}$:

$$1 \to \mathbb{Z} \to \pi_2(\mathbb{R}P^2, \mathbb{R}P^1) \to \mathbb{Z} \to 1. \tag{1}$$

This description of $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1)$ would be sufficient for this exercise, but we can in fact show that the extension is trivial (Exercise 4.2.11 in Hatcher's book). If we can show that the action $\mathbb{Z} \curvearrowright \mathbb{Z}$ defined by (1) is trivial, it would imply that $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1) \cong \mathbb{Z} \times \mathbb{Z}$. Since the action is defined by conjugation of $\mathbb{Z}$ in $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1)$, it is sufficient to show that $\mathbb{Z}$ lies in the center of $\pi_2(\mathbb{R}P^2, \mathbb{R}P^1)$. This is argued by a picture in this post on MathStackExchange. □

◊Exercice 2. Some properties of relative homotopy groups.

1. If (A, a) is a pointed space, what are $\pi_n(A, A, a)$ and $\pi_n(A, a, a)$ for $n \geq 1$?
2. If $A \subset X$ is a homotopy equivalence, show that $\pi_n(X, A, a) = 0$ for $a \in A$ and $n \geq 1$.
3. If (X, A, a) is a pointed pair where X is contractible, what can you say about $\pi_n(X, A, a)$ for $n \geq 1$?
4. For a pair (X, A) of path-connected spaces and $a \in A$, show that $\pi_1(X, A, a)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, a)$ represented by loops in A based at a.
5. Show that in general, it is not possible to find a group structure on $\pi_1(X, A, a)$ such that $\pi_1(X, a) \to \pi_1(X, A, a)$ is a morphism of groups.

Proof. 1. The long exact sequence for pairs shows that $\pi_n(A, A, a) = 0$ and $\pi_n(A, a, a) \cong \pi_1(A, a)$ for all $n \geq 1$.

2. By exercise 2.1 of sheet 3, we know that $A \subset X$ is a weak homotopy equivalence. The long exact sequence yields the result.

3. Again, the long exact sequence yields $\pi_n(X, A, a) \cong \pi_{n-1}(A, a)$ for all $n \geq 1$.

4. Using that $\pi_0(A) = 0$, the long exact sequence yields a short exact sequence of sets

$$1 \to H \to \pi_1(X, a) \to \pi_1(X, A, a) \to 1.$$

This yield the desired set bijection. Explicitly, to a map $\alpha : (D^1, S^0) \to (X, A)$ (that is a path in X starting at a and endpoint in A), we associate a loop $s(\alpha) := \alpha \cdot \gamma_\alpha : I \to X$ where γ_α is a chosen path in A from $\alpha(1) \in A$ to $a \in A$ (using path connectedness of A). Then the mapping $\pi_1(X, A, a) \to \pi_1(X, x)/H$ defined by $\alpha \mapsto [s(\alpha)]H$ is a well defined set bijection (for you to check).

5. Let $S^1 \subset S^1 \vee S^1$ be the inclusion of a circle into a wedge. The long exact sequence of the pair ends with

$$\pi_2(X, A, a) \xrightarrow{0} \langle a \rangle \hookrightarrow \langle a, b \rangle \xrightarrow{j_*} \pi_1(X, A, a) \to 1.$$

If $\pi_1(X, A, a)$ had a compatible group structure, it would imply that $\langle a \rangle = ker(j_*)$ is a normal subgroup of $\langle a, b \rangle$, which is not true.

□

Exercice 3. An H-cogroup structure in pointed pairs.
Recall that the category $Top_*^{(2)}$ of pointed pairs has objects (X, A, a) where $a \in A \subset X$ and morphisms $(X, A, a) \to (Y, B, b)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$ and $f(a) = b$.

Given a pointed pair (X, A, a), show that the group structure on $\pi_n(X, A, a)$ for $n \geq 2$ is induced by an H-cogroup structure on $(D^n, S^{n-1}, *)$ in the category $Top_*^{(2)}$ of pairs of pointed spaces.

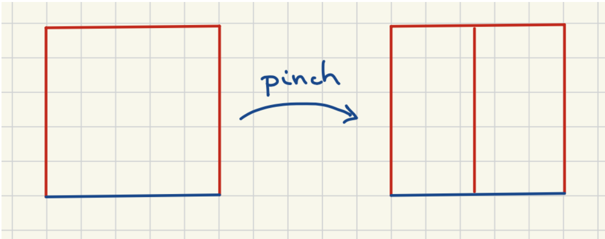
Observe first that the pinch map is not a map of pointed spaces when $n = 1$. It would correspond to identify the "equator" of $D^1 \approx [-1; 1]$ which should become the base point, but the base point of S^0 is 1...
When $n \geq 2$ this works. Let us write $D^{n-1} \subset D^n$ for the subspace of the n-disc consisting of all points whose last coordinate is zero (for the disc D^2 in $\mathbb{R}^2$ this is just the interval on the Ox axis). The intersection of D^{n-1} with S^{n-1}, the boundary of D^n is its equator, and it is not empty since $n \geq 2$. Therefore the pinch map that collapses D^{n-1} induces a pointed map of pairs

$$(D^n, S^{n-1}) \to (D^n \vee D^n, S^{n-1} \vee S^{n-1})$$

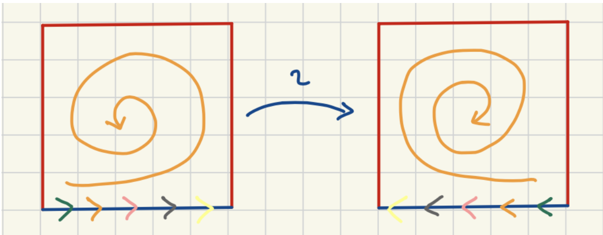
We claim this is a co-H-group in the category of pointed pairs. The map to the base point $(1, 1)$, seen as a pair of identical singletons provides a neutral element. The same homotopy as in the absolute case can be used to show that, and the associativity goes through in the same way.

We deal with the inverse in the $n = 2$ case, and then suspension provides the right map for larger $n > 2$. Instead of D^2 we define ι on I^2, where we view D^2 as this square with the boundary except

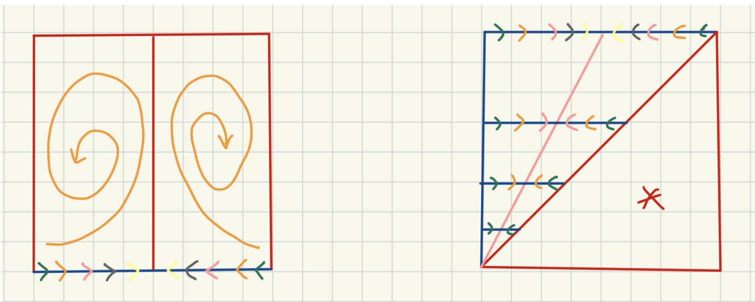
the bottom $I \times 0$ collapsed (we indicate this in red on the pictures below). This bottom segment is then identified with the boundary circle of the disc.
The comultiplication (pinch) map is given by collapsing the segment $1/2 \times I$:



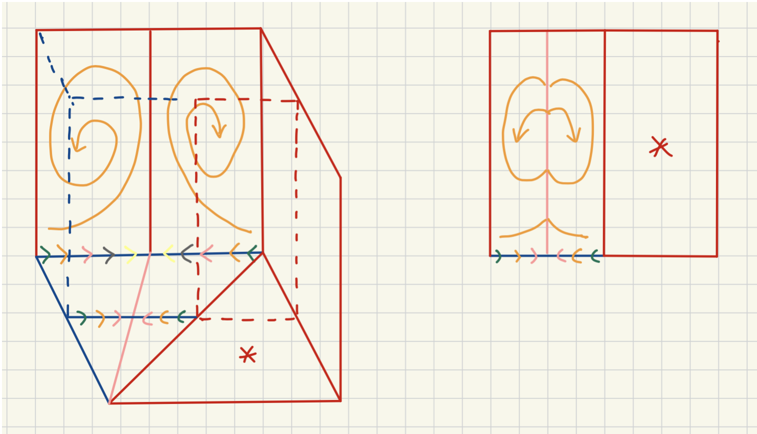
The inverse is the reflection along this same axis, the multicolored arrows on the bottom segment show the direction of the loop along the boundary and the decorative spiral illustrates the orientation of the square:



Therefore the axiom for the inverse we have to verify is whether the composition of the pinch map, followed by the identity wedge the inverse, and finally by the fold map, is null-homotopic, as a map of pairs. We see below on the left the effect of this triple composition on the original square On the bottom segment $I \times 0$ the situation is the one we know from the absolute case, the picture below on the right explains one possible nullhomotopy:



To conclude we have to extend this nullhomotopy to the whole cube indicated below on the left. It contains the above nullhomotopy on the bottom square and should start in the back with the triple composition, and end with a constant map to the base point. This homotopy is hard to draw entirely, so we simply indicate on the right a vertical slice at the dashed position on the left:



Exercice 4. Extending the sequence of a pair.

1. For a pointed pair (X, A, a), show that the sequence $\pi_1(X, a) \to \pi_1(X, A, a) \xrightarrow{\partial} \pi_0(A, a) \to \pi_0(X, a)$ is exact.

2. How can you define $\pi_0(X, A, a)$ so that $\cdots \to \pi_0(A, a) \to \pi_0(X, a) \to \pi_0(X, A, a) \to 0$ is exact?

Proof. 1. Recall that exactness for sets means that kernel and image coincide, where we choose a base point in each of these sets instead of a neutral element. This base point is always the constant map to the base point $a \in A \subset X$. Then kernel means the subset of those elements whose image is the class of this constant map c_a.

To compare kernels with images it is convenient to use the description of $\pi_1(X; a)$ not as loops, but paths $D^1 \to X$ such that the boundary S^0 is sent to the base point a. The map j_* is then simply induced by the inclusion of pairs $(X, a) \subset (X, A)$.

2. One could define $\pi_0(X, a; a)$ as $\pi_0(X/A; \bar{A})$. The map collapsing A makes a single path component out of all those of X containing a point in A. It is clearly surjective and its kernel consists precisely of the components of X intersecting A.

□

◊Exercice 5. The long exact sequence of a triple.
A pointed triple (X, A, B, b) consists of spaces X, A, B with $B \subset A \subset X$ and a basepoint $b \in B$. For each $n \geq 1$, define a 'boundary' map $\partial : \pi_n(X, A, b) \to \pi_{n-1}(A, B, b)$ as the composite $\pi_n(X, A, b) \xrightarrow{\delta} \pi_{n-1}(A, b) \xrightarrow{i} \pi_{n-1}(A, B, b)$ where δ is the connecting map in the sequence for (X, A) and i is induced by the inclusion.

Show that there is a long exact sequence

$$\cdots \to \pi_n(A, B, b) \to \pi_n(X, B, b) \to \pi_n(X, A, b) \xrightarrow{\partial} \pi_{n-1}(A, B, b) \to \cdots \to \pi_1(X, A, b).$$

Hint. You can write down three long exact sequences of pairs and put them together in the form of a braided diagram. The rest is formal so you don't have to use the explicit formulas.

Proof. Check out this video `https://www.youtube.com/watch?v=z5gWuOmUezs` where the long exact sequence of a triple is proved for homology. The same proof works for the long exact sequence in homotopy. For a computational proof, check out Theorem 4.3 in Hatcher's book. □

◊Exercice 6. Higher relative homotopy groups are abelian.
Let I^n be the n-cube and $J^{n-1} \subset \partial I^n = I^{n-1} \times \partial I \cup \partial I^{n-1} \times I$ be the subset $I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I$. For triples (X, A, B) and (X', A', B'), denote $[(X, A, B), (X', A', B')]$ the set of homotopy classes of maps of triples, ie. maps $f : X \to X'$ such that $f(A) \subset A'$ and $f(B) \subset B'$.
Given a subspace $A \subset X$ and a basepoint $a \in A$, show that $\pi_n(X, A, a) \cong [(I^n, \partial I^n, J^{n-1}), (X, A, \{a\})]$. We can hence represent elements in $\pi_n(X, A, a)$ by maps $I^n \to X$. For $\alpha, \beta : I^n \to X$, define a map $\alpha *_i \beta : I^n \to X$ by the formula

$$(\alpha *_i \beta)(t_1, \cdots, t_n) = \begin{cases} \alpha(t_1, \cdots, 2t_i, \cdots, t_n) & \text{if } 0 \le t_i \le \frac{1}{2} \\ \beta(t_1, \cdots, 2t_i - 1, \cdots, t_n) & \text{if } \frac{1}{2} \le t_i \le 1 \end{cases}$$

1. If $n \ge 2$ and $i < n$, show that $*_i$ defines a group structure on $\pi_n(X, A, a)$.
2. If $i, j < n$ with $i \ne j$, show that $*_i$ and $*_j$ satisfy the interchange law:
$$(\alpha *_i \beta) *_j (\gamma *_i \delta) = (\alpha *_j \gamma) *_i (\beta *_j \delta).$$
3. Use the Heckmann-Hilton argument to show that $\pi_n(X, A, a)$ is an abelian group when $n \ge 3$.
4. Find an inclusion of spaces $A \subset X$ for which $\pi_2(X, A, a)$ is not abelian.

Proof. □

Exercice 7. The action of π_1 on π_n.
Let (X, x) be a path-connected pointed space and γ a path in X with endpoints $x, y \in X$.

1. Use γ to define an isomorphism $\varphi_\gamma : \pi_n(X, x) \cong \pi_n(X, y)$ for $n \ge 1$.
__Hint__: you can represent a class in $\pi_n(X, x)$ by a map $(D^n, S^{n-1}) \to (X, a)$.
2. Restricting to loops based at x, show that $\gamma \mapsto \varphi_\gamma$ defines an action of $\pi_1(X, x)$ on $\pi_n(X, x)$.
3. Show that this action is induced from a map $S^n \to S^n \vee S^1$ by an application of the functor $[-, X]_*$.
__Hint__: use the fact that $S^n \cong D^n/S^{n-1}$.
4. * Define similarly an action of $\pi_1(A, a)$ on $\pi_n(X, A, a)$ for $n \ge 1$, and show that it is induced from a map of pairs by application of the functor $[-, (X, A)]_*$.

*Proof*1 & 2. A nice explicit and visual description is on pages 341-342 in Hatcher's book, using the model $(I^n, \partial I^n)$ for (D^n, S^{n-1}). One could also describe it directly using the next point.

3. Choose $(0, \dots, 1) \in S^n$ as the base point of the n-sphere. Define a map $f : S^n \to S^n \vee I$ by

$$(x_0, \dots, x_n) \mapsto \begin{cases} \frac{(x_0, \dots, x_{n-1}, 2x_n)}{\|(x_0, \dots, x_{n-1}, 2x_n)\|} \in S^n & \text{if } 0 \le x_n \le 1/2; \\ 2x_n - 1 \in I & \text{if } 1/2 \le x_n \le 1. \end{cases}$$

Notice that f takes the base point to the endpoint of the interval. This map is a homotopy equivalence with inverse $S^n \vee I \to S^n$ which contracts the interval to the base point of S^n. Given a path γ in X from x to y, one obtains the base point change isomorphism $\pi_n(X,x) \to \pi_n(X,y)$ defined by $g \mapsto (g \vee \gamma) \circ f$. When we want to restrict to based loops in X, one considers the composition $S^n \xrightarrow{f} S^n \vee I \to S^n \vee S^1$, which becomes a based map. Since representable functors preserves limits, $[-,X]_* : hTop_*^{op} \to Set$ turns wedge sums into products. It follows that $S^n \to S^n \vee S^1$ induces the action $\pi_n(X,x) \times \pi_1(X,x) \to \pi_n(X,x)$. $\square$

$\Diamond$ indicates the weekly assignments.