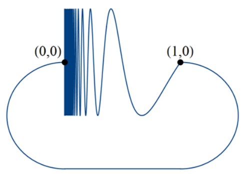
EPFL - Autumn semester 2022-2023 J. Scherer
Homotopy theory **Sheet 3**
Mathematics 4th October 2022

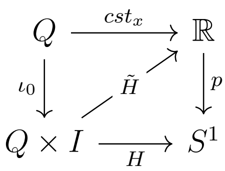
Exercice 1. The 'quasi-circle'.
Define the 'quasi-circle' to be a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph $y = \sin(\frac{\pi}{x})$, the segment $[-1, 1]$ in the y-axis and an arc connecting these two pieces (see picture).
Show that the quasi-circle has trivial homotopy groups, but is not contractible.



Proof. Let $\Gamma = \{(x, \sin(\frac{\pi}{x})) \mid 0 < x \leq 1\}$ be the portion of the graph, L be the segment $[(0, -1), (0, 1)]$ and C be the arc joining $(0, 0)$ and $(1, 0)$ containing these two points. We have $Q = \Gamma \cup L \cup C$.

Since there is no path from Γ to L without going through C, and because S^n is compact and connected, any map $\varphi : S^n \to X$ has to avoid a point in Γ, say $x \in \Gamma$. But then $Q \setminus \{x\}$ has two path connected components which are contractible, so φ has its image in one of these components and so has to be nullhomotopic. This proves that $\pi_n(Q) = 0$ for all $n \in \mathbb{N}$.

To show that Q is not contractible, consider the quotient map $f : Q \to Q/L \simeq S^1$ which collapses the segment L. We claim that f is *not* nullhomotopic. This proves that Q is not contractible since any map with contractible domain is nullhomotopic. Suppose for a contradiction that H is a nullhomotopy $cst_{x_0} \simeq f$. Consider the universal cover $p : \mathbb{R} \to S^1$ and a lift $y_0 \in \mathbb{R}$ of x_0. Since any covering space is a fibration, we can lift the nullhomotopy H to a nullhomotopy $\tilde{H} : Q \times I \to \mathbb{R}$:



Since $\tilde{f} := \tilde{H}_1$ lifts f which collapses the segment L and is injective elsewhere, the lift $\tilde{f}$ is injective, except maybe on the segment L. But the image of L has to be contained in the fiber $\mathbb{Z}$ of p which is discrete, and thus $\tilde{f}(L)$ is a point. The map factors through the quotient by an injective map $\tilde{\tilde{f}} : S^1 \simeq Q/L \to \mathbb{R}$. But by the intermediate value theorem, there can't exists injective maps $S^1 \to \mathbb{R}$, a contradiction. □

◊Exercice 2. Weak equivalences vs. homotopy equivalences
Recall that a weak equivalence is a map $f : X \to Y$ that induces isomorphisms on all homotopy groups $f_* : \pi_n(X, x) \cong \pi_n(Y, f(x))$ for all n and all base points $x \in X$. The objective of this exercise is to give an idea of how to construct a space weakly equivalent to the circle, but which is not homotopy equivalent.

1. Show that a homotopy equivalence is a weak equivalence.

2. Consider the finite space X with four points a, b, c, d whose topology is given by the following list of open subsets: $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X$. Show that X is path connected.

3. Construct a surjective (continuous) map $f\colon S^1 \to X$.

4. Show that the only (continuous) maps $X \to S^1$ are constant.

5. Show that the open subspace $\{a, b, c\} \subset X$ is contractible. This shows that X can be seen as a union of two contractible open subspaces whose intersection is a discrete subspace with two points.

Proof. 1. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. We fix an arbitrary base point $x_0 \in X$ and show that $\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all n. Note that the homotopy $gf \simeq 1_X$ is not necessarily pointed and $gf(x_0) \neq x_0$, so it is not true that $[gf] = [1_X] \in [X, X]_*$. However we can use a path $x_0 \to gf(x_0)$ to remedy this problem. We use the following claim:

Claim. Let $h : X \to X$ be homotopic to the identity, then $\pi_n(h) = \varphi_\gamma : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, h(x_0))$ is the base point change isomorphism given in exercise 7 of sheet 4, where $\gamma = H|_{\{x_0\}\times I}$ is the path $h(x_0) \to x_0$ given by the homotopy $H : h \simeq 1_X$.

Proof. To see this, let $f : S^n \to X$ be a pointed map, and write γ_t the reparametrized path $h(x_0) \to \gamma(t)$. Then $\varphi_{\overline{\gamma_t}}(H_t \circ f)$ is a based homotopy $h \circ f \simeq_* \varphi_\gamma(f)$. This implies that $[h \circ f] = [\varphi_\gamma(f)]$ in $\pi_n(X, h(x_0))$ as desired. □

Remark. When $n = 1$, we can recover the proof that a homotopy equivalence induces an isomorphism on π_1, where the map $\varphi_\gamma : \pi_1(X, x_0) \to \pi_1(X, h(x_0))$ is given by conjugation $\lambda \mapsto \gamma \cdot \lambda \cdot \overline{\gamma}$

But now consider the following diagram:

$$\pi_n(X, x_0) \xrightarrow{\pi_n(f)} \pi_n(X, f(x_0)) \xrightarrow{\pi_n(g)} \pi_n(X, gf(x_0)) \xrightarrow[\pi_n(f)]{} \pi_n(X, fgf(x_0))$$
$$\pi_n(X, x_0) \xrightarrow{\pi_n(gf)} \pi_n(X, gf(x_0)), \qquad \pi_n(X, f(x_0)) \xrightarrow{\pi_n(fg)} \pi_n(X, fgf(x_0))$$

The claim shows that the two curved arrows are isomorphisms. This implies by direct inspection that $\pi_n(f)$ is an isomorphism. This last implication is called the 2-out-of-6 property.

2. The map $\gamma : I \to X$ given by $t \mapsto \begin{cases} a & t < 1; \\ d & t = 1 \end{cases}$ is continuous and is a path from a to d in X. By symmetry of the topology on X, we obtain a path from a to c, and similarly from b to c.

3. Let $f : I \to X$ be the concatenation of paths $a \to b \to c \to d \to a$, which factors through a surjective map $S^1 \to X$.

4. Let $f : X \to S^1$ be a continuous map. Since X is path connected, its image $f(X)$ is path connected as well. But $f(X) \subset S^1$ is finite so it must be a point.

5. The open space $\{a, b, c\}$ can be contracted to c using the homotopy $H : \{a, b, c\} \times I \to \{a, b, c\}$ defined by $(x, t) \mapsto \begin{cases} x & t < 1; \\ c & t = 1. \end{cases}$ A direct inspection shows that H is continuous.

□

◊Exercice 3. Higher homotopy groups are abelian.

1. Prove that the fundamental group of any H-space (not necessarily homotopy associative) is abelian.

2. Prove (again) that for any space X the homotopy groups $\pi_n(X)$ are abelian for $n \geq 2$.

Proof. 1. Let X be an H-space with identity $e \in X$ and multiplication $\mu : X \times X \to X$. Let $p, q : I \to X$ be loops in X at e. We want a homotopy $p \star q \simeq q \star p$. Define $F : I \times I \to X$ by the formula $(s,t) \mapsto \mu(p(s), q(t))$. Then $H(-,0) = H(-,1) = \mu(p,e) \simeq p$ and $H(0,-) = H(1,-) = \mu(e,q) \simeq q$. Now let $\gamma : I \to I \times I$ be the path defined by $\gamma(t) = (2t, 0)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma(t) = (1, 2t-1)$ for $\frac{1}{2} \leq t \leq 1$. Let $\gamma' : I \to I \times I$ be the path defined by $\gamma(t) = (0, 2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma(t) = (2t-1, 1)$ for $\frac{1}{2} \leq t \leq 1$. Let $H : I \times I \to I \times I$ be a homotopy $\gamma \simeq \gamma'$. Then $F \circ H$ is a homotopy $p \star q \simeq q \star p$. (Make a drawing to visualize it).

2. If $n \geq 2$, then $\pi_n(X) = [S^n, X]_* \cong [S^1 \wedge S^{n-1}, X]_* \cong [S^1, Map_*(S^{n-1}, X)]_* \cong \pi_1(\Omega^{n-1}X)$ and $\Omega^{n-1}X$ is an H-space, so by 1) its fundamental group is abelian.

□

◊Exercice 4. Homotopy groups and coverings.

1. Show that a covering space projection $p : E \to B$ induces isomorphisms $p_* : \pi_n(E) \cong \pi_n(B)$ for any $n \geq 2$ and any choice of basepoint $e \in E$.

2. Compute $\pi_n(S^1)$ for all $n \geq 1$.

3. Compute $\pi_n(K)$ for all $n \geq 1$, where K is the Klein bottle.

4. Compare the higher homotopy groups of $\mathbb{R}P^2$ with those of S^2.

Proof. 1. The lifting property of covering spaces says that given a map $f : X \to B$ where X is a path-connected and locally path-connected space with base point $x \in X$, f admits a lifting $\overline{f} : X \to E$ (such that $p \circ \overline{f} = f$ if and only if $f_*(\pi_1(X,x)) \subseteq p_*(\pi_1(E,e))$. Furthermore, if we require the lifting to be pointed, i.e $f(x) = e$, the the lifting is unique. For $X = S^n$, the first part gives us surjectivity of p_*, while the second part gives us injectivity.

2. We know that there is a covering projection $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is contractible, we conclude that $\pi_n(S^1)$ is trivial for $n \geq 2$. It is common knowledge that $\pi_1(S^1) \cong \mathbb{Z}$.

3. We know that there is a covering projection $\mathbb{R}^2 \to K$. Since $\mathbb{R}^2$ is contractible, we conclude that $\pi_n(K)$ is trivial for $n \geq 2$. Seifert Van Kampen shows that $\pi_1(K) \cong \langle a, b | abab^{-1} = 1 \rangle$.

4. We know that there is a covering projection $p : S^2 \to \mathbb{R}P^2$. We conclude that $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2)$ for $n \geq 2$. Moreover we know that the fiber of p is in bijection with the index of the fundamental group of S^2 in $\mathbb{R}P^2$. Since the fiber is of cardinality 2, it implies that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$.

□

Exercice 5. Homotopy groups of products.

1. Given a collection of path-connected spaces X_α, show that there are isomorphisms $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for any choices of basepoints $x_\alpha \in X_\alpha$.

2. Compute $\pi_n(T)$ for all $n \geq 2$, where $T = S^1 \times S^1$ is the torus (you can also use Exercise 4).

Proof. 1. We have seen in Sheet 1. that $Map(X, \prod_\alpha Y_\alpha) \cong \prod_\alpha Map(X, Y_\alpha)$ when X is Hausdorff. Hence we find $Map_*(S^n, \prod_\alpha X_\alpha) \cong \prod_\alpha Map_*(S^n, X_\alpha)$. Applying π_0, which commutes with products, we obtain $[S^n, \prod_\alpha X_\alpha]_* \cong \prod_\alpha [S^n, X_\alpha]_*$ as desired.

2. $\pi_n(S^1 \times S^1) \cong \pi_n(S^1) \times \pi_n(S^1) \cong \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq 2; \\ \mathbb{Z} \times \mathbb{Z} & \text{if } k = 1. \end{cases}$

□

◊Exercice 6. co-H-groups.

1. Let X be a co-H-group. Show that $[X, -]_*$ defines a functor from pointed topological spaces to the category of groups.

2. Let X be a pointed space such that $[X, -]_*$ defines a functor from pointed topological spaces to the category of groups. Show that X is a co-H-group.

3. Show that a co-H-map $X \to X'$ between co-H-groups induces a group homomorphism $[X', Y]_* \to [X, Y]_*$ for any pointed space Y.

Proof. It is a special case of the next exercise, for $\mathcal{C} = hTop_*^{op}$. □

◊Exercice 7. Group objects in categories. Let $\mathcal{C}$ be a category with products and a terminal object I.

1. By analogy with the definition of H-group, define the notion of *group object* in $\mathcal{C}$ so that group objects in *Sets* are groups.

2. Show that an object G in $\mathcal{C}$ is a group object if and only if $mor_{\mathcal{C}}(-, G) : \mathcal{C}^{op} \to Sets$ factors through the category of groups. (It is the if part which takes more work as you will have to identify a neutral element, a multiplication, and an inverse).

3. Identify all group objects in the category of groups.

Proof. 1. Check out the nLab page *group object* for the definition.

2. We work in a more general setting. Let $F : \mathcal{D} \to \mathcal{D}'$ be a finite product preserving functor between categories with finite products. It is immediate to see that F preserves group objects and group homomorphisms. Stated differently, F induces a functor $Grp(F) : Grp(\mathcal{D}) \to Grp(\mathcal{D}')$. Since the Yoneda embedding $\mathcal{C} \to Set^{\mathcal{C}^{op}}$ preserves finite products (limits in general), we obtain that $mor_{\mathcal{C}}(-, G)$ is a group object in $Set^{\mathcal{C}^{op}}$. This is easily seen to factor through the category of groups.

   Conversely suppose that F as above is fully faithful. We can again show that F reflects group objects. For example, if $m : F(C) \times F(C) \to F(C)$ is a multiplication map, we can lift it through $\mathcal{D}(G \times G, G) \cong \mathcal{D}'(F(G \times G), F(G)) \cong \mathcal{D}'(F(G) \times F(G), F(G))$ to a multiplication $m' : G \times G \to G$ such that $F(m') = m$. This multiplication m' is associative if and only if

m is by naturality of the isomorphism $\mathcal{D}(-,-) \cong \mathcal{D}'(F-,F-) : \mathcal{C}^{op} \times \mathcal{C} \to Set$. Similarly we can lift the neutral element, as well as the inverse map, which have the desired properties by naturality. The result follows by the fact that the Yoneda embedding is fully faithful (by the Yoneda lemma), and that $mor_{\mathcal{C}}(-, G)$ is a group object in $Set^{\mathcal{C}^{op}}$ if and only if it factors through Grp.

3. Let G be a group with multiplication $\star$. Suppose furthermore that G is a group object in Grp, with multiplication $\circ$. Since $\circ : G \times G \to G$ is a morphism in Grp, it must satisfy $\circ((g,g') \star (h,h')) = \circ(g,g') \star \circ(h,h')$. The former is $\circ(g \star h, g' \star h') = (g \star h) \circ (g' \star h')$ while the latter is $(g \circ g') \star (h \circ h')$. This is exactly the interchange law, which implies that the two group operation agree and are commutative by the Eckmann-Hilon argument. We conclude that group objects in Grp are exactly abelian groups.

$\square$

Exercice 8. The category of pairs of spaces.
Write $Top_{(2)}$ for the category of pairs of spaces. Objects are pairs (X, A) where $A \subset X$ and morphisms $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$.
The set $Hom_{\mathrm{Top}_{(2)}}\big((X, A), (Y, B)\big)$ is topologized as a subspace of $Map(X, Y)$. The resulting space is denoted $Map_{(2)}\big((X, A), (Y, B)\big)$.

1. Show that $Top_{(2)}$ is indeed a category.

2. Show that the forgetful functor $Top_{(2)} \to Top$, $(X, A) \mapsto X$ has both a left and a right adjoint.

For pairs $(X, A), (Y, B)$, define $(X, A)\square(Y, B) = (X \times Y, X \times B \cup A \times Y)$.

3. Prove the exponential law for pairs of locally compact Hausdorff spaces :

$$Map_{(2)}\big(\, (X,A)\square(Y,B), (Z,C)\,\big) \cong Map_{(2)}\Big(\, (X,A), \big(\, Map_{(2)}((Y,B),(Z,C)), Map(Y,C)\,\big)\Big)$$

Note that there is an inclusion functor $Top_* \hookrightarrow Top_{(2)}$ given by $(X, x) \mapsto (X, \{x\})$.

4. Show that the formula $q(X, A) = (X/A, *)$ defines a functor $Top_{(2)} \to Top_*$.

5. Show that q is left adjoint to the inclusion $Top_* \hookrightarrow Top_{(2)}$. Is it an equivalence of categories?

Proof. 1. Check the axioms.

2. You can easily verify that the left adjoint is given by $L(X) = (X, \emptyset)$, while the right adjoint is given by $R(X) = (X, X)$.

3. The LHS of the homeomorphism is a subspace of the LHS of the usual exponential law

$$Map(X \times Y, Z) \cong Map(X, Map(Y, Z)).$$

Under this homeomorphism, a direct inspection shows that the LHS corresponds precisely to the RHS of the exponential law for pairs.

4. The functor is well defined on morphisms because for a map $f : X \to Y$ such that $f(A) \subseteq B$, the composition $X \to Y \to Y/B$ factors through X/A. Then check that q preserves composition and identities.

5. The universal property of the quotient states that

$$Hom_{Top_*}((X/A, *), (Y, B)) \cong Hom_{Top_{(2)}}((X, A), (Y, y))$$

as desired.

$\square$

Exercice 9. Cylinder, cone, suspension and their reduced variants.
Given a space X, write $Cyl(X) = X \times I$, $CX = (X \times I)/X \times \{1\}$ and $SX = (X \times I)/(X \times \partial I)$ for the cylinder, cone and suspension of X respectively.
Given a pointed space X, write $\overline{Cyl}(X) = X \rtimes I$, $\overline{C}X = (X \rtimes I)/X \times \{1\}$ and $\Sigma X = (X \rtimes I)/(X \times \partial I)$ for the reduced versions of the above.

1. Show that Cyl, C, S are functors $Top \to Top$ and that $\overline{Cyl}, \overline{C}, \Sigma$ are functors $Top_* \to Top_*$.
2. Prove that there are natural transformations $Cyl \to C \to S$ and $\overline{Cyl} \to \overline{C} \to \Sigma$.
3. Prove that the following diagram of functors $Top \to Top$ is commutative

$$\begin{array}{ccccc} CylU & \longrightarrow & CU & \longrightarrow & SU \\ \downarrow & & \downarrow & & \downarrow \\ U\overline{Cyl} & \longrightarrow & U\overline{C} & \longrightarrow & U\Sigma \end{array} .$$

Here $U : Top_* \to Top$ denotes the forgetful functor.

Proof. It is straightforward to check the three points. The natural transformations $Cyl \to C \to S$ are given by the quotient maps $X \times I \to (X \times I)/X \times \{1\} \to (X \times I)/(X \times \partial I)$ and the reduced case is analogous. In a similar fashion, the vertical transformations in the diagram are given by the quotient of $Cyl(X), CX, SX$ by $\{x\} \times I$. $\square$

$\lozenge$ indicates the weekly assignments.