EPFL - Autumn semester 2022-2023	J. Scherer
Homotopy theory	**Sheet 2**
Mathematics	26th September 2022

◊Exercice 1. Homotopies are paths in a function space.
A *homotopy* between maps $f, g : X \to Y$ is a map $H : X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. Write $H : f \simeq g$ in such a situation. Given a space X and $x, y \in X$, a path in X between x and y is a (continuous) map $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The set of homotopy classes between X and Y, denoted $[X, Y]$, is the set of equivalences classes of maps $X \to Y$ under the relation $\simeq$.

1. If X is locally compact Hausdorff, show that there is a bijection between the set of homotopies $f \simeq g$ and the set of paths in $Map(X, Y)$ between f and g.
2. Under the same hypotheses, show that $[X, Y] \cong \pi_0 Map(X, Y)$.

Proof.

1. Since I is Hausdorff and X is assumed to be locally compact Hausdorff, there is a homeomorphism $Map(X \times I, Y) \cong Map(I, Map(X, Y))$, which restricts to a bijections of the underlying sets. Moreover, one checks that given a homotopy $h : f \simeq g$, the adjoint map $a(h) : I \to Map(X, Y)$, which is defined by $a(h)(t) = h(-,t)$ satisfies $a(h)(0) = f$ and $a(h)(1) = g$.
2. Under the bijection proved in 1), two maps $f, g : X \to Y$ are homotopic iff they are in the same path component of $Map(X, Y)$, so that the two equivalence relations $\simeq$ and 'being in the same path component' are the same on $Map(X, Y)$, hence the quotient sets are in bijection.

□

Exercice 2. Composition preserves and detects homotopy equivalences.
Let X, Y, Z be locally compact spaces.

1. If $g, g' : Y \to Z$ are homotopic maps, show that $g \circ -$ and $g' \circ -$ are homotopic maps $Map(X, Y) \to Map(X, Z)$.
2. Show that if $g : Y \to Z$ is a homotopy equivalence, then so is $g \circ -$.
3. Suppose that $g \circ -$ is a homotopy equivalence $Map(X, Y) \to Map(X, Z)$ for any space X. Show that g is a homotopy equivalence.

Proof.

1. If $h : Y \times I \to Z$ is a homotopy $g \simeq g'$, then $H : Map(X, Y) \times I \to Map(X, Z)$ defined by $H(f, t) = h(-, t) \circ f$ is a homotopy $(g \circ -) \simeq (g' \circ -)$.
2. Since g is a homotopy equivalence, it has a homotopy inverse $f : Z \to Y$ and we have homotopies $g \circ f \simeq id_Z$ and $f \circ g \simeq id_Y$. We have to prove that $f \circ -$ is a homotopy inverse for $g \circ -$. By 1) we have that $(f \circ -) \circ (g \circ -) = (f \circ g \circ -) \simeq (id_Y \circ -) = id_{Map(X,Y)}$. Similarly $(g \circ -) \circ (f \circ -) = (g \circ f \circ -) \simeq (id_Y \circ -) = id_{Map(X,Z)}$.
3. Taking $X = \{*\}$ and under the homeomorphisms $Map(*, Y) \cong Y$ and $Map(*, Z) \cong Z$, the map $g \circ -$ is identified with g, so that g is a homotopy equivalence.

□

$\Diamond$**Exercice 3. The half-smash product.**
Let (X, x) be a pointed space and Y be an unpointed space. Define the half-smash product $X \rtimes Y$ to be the collapse $(X \times Y)/(\{x\} \times Y)$, with base point $\{x\} \times Y$. For an unpointed space Y, define $Y_+ = Y \coprod \{*\}$ to be Y with a disjoint basepoint.

1. Prove that $X \rtimes Y \cong X \wedge Y_+$.

2. Prove that for a pointed space Y there is a homeomorphism

$$Map_*(X_+, Y) \cong Map(X, Y).$$

3. For spaces X, Y with X unpointed and Y pointed, prove the adjunction identity

$$Map_*(X \rtimes Y, Z) \cong Map(Y, Map_*(X, Z)).$$

4. Suppose X, Y, Z are pointed spaces with Y locally compact. Prove that

$$Map_*(X \rtimes Y, Z) \cong Map_*(X, Map(Y, Z)).$$

Proof. 1. We have

$$\begin{aligned} X \wedge Y_+ &= (X \times Y_+)/(\{x\} \times Y_+ \cup X \times \{*\}) \\ &= ((X \times Y) \sqcup X)/(\{x\} \times Y \sqcup X) \\ &= (((X \times Y) \sqcup X)/X)/(\{x\} \times Y) \\ &= (X \times Y)/(\{x\} \times Y) = X \rtimes Y. \end{aligned}$$

2. Define a map $\varphi : Map_*(X_+, Y) \to Map(X, Y)$ by $\varphi(f) = f_{|X}$. This map is a homeomorphism since $Map_*(X_+, Y)$ is a subspace of $Map(X_+, Y)$ and under the homeomorphism $Map(X_+, Y) \cong Map(X, Y) \times Y$ it is identified with the composite $Map(X, Y) \to Map(X, Y) \times Y \to Map(X, Y)$ which is the identity.

3. Using 1) and 2) we have

$$\begin{aligned} Map_*(X \rtimes Y, Z) &\cong Map_*(X \wedge Y_+, Z) \\ &\cong Map_*(Y_+, Map_*(X, Z)) \\ &\cong Map(Y, Map_*(X, Z)). \end{aligned}$$

4. Similarly, we find

$$\begin{aligned} Map_*(X \rtimes Y, Z) &\cong Map_*(X \wedge Y_+, Z) \\ &\cong Map_*(X, Map_*(Y_+, Z)) \\ &\cong Map_*(X, Map(Y, Z)). \end{aligned}$$

$\square$

Exercice 4. Some special pullbacks.

1. Let $f : X \to Y$ be a map and $y \in Y$. Show that the subspace $f^{-1}(y) \subseteq X$ defined by $\{x \in X \mid f(x) = y\}$ makes the following square a pullback of spaces:

$$\begin{array}{ccc} f^{-1}(y) & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{y\} & \hookrightarrow & Y \end{array} .$$

2. Let $f : X \to Y$ be a map and $A \subseteq Y$. Define a subspace $B \subseteq X$ by $\{x \in X \mid f(x) \in A\}$. Find the map $g : B \to A$ that makes the following square a pullback of spaces:

$$\begin{array}{ccc} B & \hookrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \hookrightarrow & Y \end{array} .$$

(once you have identified the map g, you also have to show that the square is a pullback)

Proof. 1. Check the universal property. In general the pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$ is easily shown to be homeomorphic to the subspace of $X \times Y$ defined by $\{(x, y) \mid f(x) = g(y)\}$.

2. Same as before. The map $g : B \to A$ is just $f_{|B}$.

□

Exercice 5. Some preservation properties of adjoint functors. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors $F \dashv G$. Show that F preserves pushouts squares and that G preserves pullback squares. For example for F, you have to show that if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array} .$$

is a pushout square in $\mathcal{C}$, then

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle F(h)} \\ F(C) & \xrightarrow[F(k)]{} & F(D) \end{array} .$$

is a pushout square in $\mathcal{D}$. By a similar argument you can show that a left adjoint preserves any colimit, and a right adjoint preserves any limit.

Proof. The usual proof : if $\{X_i \to X\}_i$ is a colimit cone in $\mathcal{C}$, and $\{F(X_i) \to Y\}_i$ is a cone in $\mathcal{D}$, then by adjunction $\{X_i \to G(Y)\}_i$ is a cone in $\mathcal{C}$, which factors through $\{X_i \to X\}_i$ by a map $\varphi : X \to G(Y)$. Then the adjunct map $a(\varphi) : F(X) \to Y$ gives a factorization $\{F(X_i) \to F(X) \to Y\}_i$ of $\{F(X_i) \to Y\}_i$ which is unique. □

◊Exercice 6. The Sierpiński space.

Denote S the Sierpiński space, defined as follows: the underlying set is $\{0,1\}$, with open sets $\varnothing, \{1\}, \{0,1\}$. If $A \subset X$ is an inclusion of spaces, denote $\chi_A : X \to \{0,1\}$ the characteristic function of A, defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

1. Given a space X, identify the set $Hom_{\mathrm{Top}}(X, S)$ of continuous maps $X \to S$.
2. For X compact Hausdorff, show that the singleton $\{\chi_X\}$ is open in $Map(X, S)$.

Denote $\mathcal{T}_X$ the topology on X. For a subset $A \subseteq X$, write $\mathcal{O}_A = \{U \in \mathcal{T}_X \mid A \subseteq U\}$.

3. Show that $\{\mathcal{O}_K \mid K \subset X \text{ compact}\}$ is the basis for a topology on $\mathcal{T}_X$. Write $\mathcal{O}(X)$ for the topology it generates and *Open*(X) for the space $(\mathcal{T}_X, \mathcal{O}(X))$.
4. Show that there is a homeomorphism $Map(X, S) \cong Open(X)$.

Proof.

1. We show that $Hom_{\mathrm{Top}}(X, S)$ is in bijection with the set $\mathcal{T}_X$ of open sets of X. Define a map $\varphi : Hom_{\mathrm{Top}}(X, S) \to \mathcal{T}_X$ by $f \mapsto f^{-1}(\{1\})$. The inverse is given by $U \mapsto \chi_U$.
2. Since X is compact, one can just say that $B(X, \{1\}) = \{\chi_X\}$ is open in $Map(X, S)$.
3. First check that the $\mathcal{O}_K$ cover $\mathcal{T}_X$: we have $U \in \mathcal{O}_\emptyset$ for any open $U \subseteq X$ since $\emptyset \subseteq X$ ic compact. Now check that for every $\mathcal{O}_K, \mathcal{O}_L$ with $K, L \subseteq X$ compact, there is a $C \subseteq X$ compact with $\mathcal{O}_C \subseteq \mathcal{O}_K \cap \mathcal{O}_L$. We find that $\mathcal{O}_K \cap \mathcal{O}_L = \mathcal{O}_{K \cup L}$ and $K \cup L$ is compact so we are done.
4. There is a bijection φ of the underlying sets by 1). This map is continuous since $\varphi^{-1}(\mathcal{O}_K) = \{\chi_U \mid U \in \mathcal{O}_K\} = \{\chi_U \mid K \subseteq U\} = B(K, \{1\})$. Moreover its inverse ψ is also continuous since $\psi^{-1}(B(K, \{1\})) = \psi^{-1}(\varphi^{-1}(\mathcal{O}_K)) = \mathcal{O}_K$.

□

Exercice 7*. Problem with locally compact spaces.

1. Show that the category of locally compact spaces is not cocomplete.
   <u>*Hint:*</u> *Consider for example the sequence of inclusions* $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$
2. Show that a colimit of locally compact spaces is compactly generated.

Proof.

1. We want to show that $\mathbb{R}^\infty = \operatorname{colim}_n \mathbb{R}^n$ is not locally compact. If it were, we could find a compact neighbourhood $K \subseteq \mathbb{R}^\infty$ of the point 0. Because $\mathbb{R}^\infty$ is metrizable, a basis of neighbourhoods of 0 consists of open balls $B(0, r)$ for $r > 0$. So $B(0, r) \subset K$ for $r > 0$ sufficiently small. But then $\overline{B}(0, \frac{r}{2}) \subseteq K$ is a closed subset of K compact, hence compact. This is a contradiction: a closed ball in $\mathbb{R}^\infty$ is not compact. To prove, say, that $\overline{B}(0, 1)$ is not compact, it suffices to prove that it is not sequentially compact. But the sequence of basis vectors $\{e_n\}_{n \geq 0}$ has no convergent subsequence.
2. Since arbitrary colimits can be formed by disjoint unions and quotients, it suffices to show that a quotient of a locally compact space is compactly generated. We prove that a quotient of a compactly generated space is still compactly generated. Since locally compact spaces are compactly generated, we get the result.

If $q : X \to Y$ is a quotient map and X is compactly generated, let kY be the space Y with the following topology $U \subseteq Y$ is k-open if and only if $U \cap K$ is open for any compact set $K \subset Y$. Since X is compactly generated, the map q induces a continuous map $\overline{q} : X \to kY$ defined by $x \mapsto q(x)$. Take $A \subseteq Y$ to be open for the k-topology. We wish to show that A is an actual open of Y. Since $\overline{q}$ is continuous we have that $\overline{q}^{-1}(A) \subseteq X$ is open. But $\overline{q}^{-1}(A) = q^{-1}(A)$. Because q is a quotient, we find that A is open in Y.

$\square$

$\lozenge$ indicates the weekly assignments.