Exercice 1. Point-set topology appetizer.
Recall that a space X is *normal* if any pair of disjoint closed subsets have disjoint neighbourhoods.

1. Show that a compact Hausdorff space is normal.

2. If X is a compact Hausdorff space covered by open subsets U_i, show that there are compact subspaces $K_i \subseteq U_i$ covering X.

Proof. 1. First prove that if $F \subseteq X$ is a closed subset and $x \notin F$, we can find disjoint open subsets $U, V \subseteq X$ such that $x \in U$ and $F \subseteq V$: for each $y \in F$, can find disjoint opens U_y, V_y with $x \in U_y$ and $y \in V_y$. Then $F \subseteq \bigcup_{y \in F} V_y$, and because F is compact, we can find a finite subcover $V = \bigcup_{i=1}^n V_{y_i} \supseteq F$. Now letting $U = \bigcap_{i=1}^n U_{y_i}$, we have that U is an open containing x. So U, V are disjoint opens with the desired properties.

Then apply the lemma to each $x \in F'$ and use the same strategy: for each $x \in F'$, can find disjoint opens U_x, V_x such that $x \in U_x$ and $F \subset V_x$. Again F' is compact and $F' \subseteq \bigcup_{x \in F} U_x$, so we can find a finite subcover $U = \bigcup_{i=1}^m U_{x_i} \supseteq F'$. Now $V = \bigcap_{i=1}^m V_{x_i}$ is an open disjoint from U and containing F.

2. We use the following fact : if X is normal and $F \subseteq X$ is closed, with open neighbourhood $F \subseteq U$, then there exists an open set $V \subseteq X$ such that $F \subseteq V \subseteq \overline{V} \subseteq U$. Indeed, U^c is closed and disjoint from F, so there are disjoint open neighbourhoods $F \subseteq V$ and $U^c \subseteq W$. Therefore $V \subseteq W^c$ and since W^c is closed, we get $\overline{V} \subseteq W^c$, hence the conclusion. Now to prove the claim, since X is compact we can assume the cover to be finite. We use induction on the number of opens. If $X = U_1 \cup U_2$, then $U_i^c \subseteq U_j$, $(i \neq j)$ so by the lemma above, there are opens V_i such that $U_j^c \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$. Since $V_1^c \cap V_2^c$ is closed and contained in the open $U_1 \cap U_2$, we can find an open W such that $V_1^c \cap V_2^c \subseteq W \subseteq \overline{W} \subseteq U_1 \cap U_2$. Now take $K_1 = \overline{V_1}$ and $K_2 = \overline{V_2} \cup \overline{W}$. They are both closed hence compact, and $K_i \subseteq U_i$. Moreover $K_1 \cup K_2 = X$. This proves the base case $n = 2$.

Now suppose the result has been proven for a cover of X by n open sets. Suppose $U_1, \cdots, U_{n+1}$ is an open cover of X. Then $U = \bigcup_{i=1}^n U_i$ and U_{n+1} cover X, so by the preceding question, we can find compact sets $K \subset U$ and $K_{n+1} \subseteq U_{n+1}$ that cover X. Now K is covered by the $U_i \cap K$, $i \leq n$. Applying the induction hypothesis to this open cover of K, we find compact sets $K_i \subset U_i \cap K$ for $i \leq n$. Then $K_1, \cdots, K_{n+1}$ are compact sets that cover X with $K_i \subseteq U_i$. □

Exercice 2. $Map(X, Y)$ inherits some topological properties of Y.

1. If Y is Hausdorff and X is any space, show that $Map(X, Y)$ is Hausdorff.

2. If X is compact and Y is metrizable, show that $Map(X, Y)$ is metrizable.

Proof. 1. Take continuous maps $f, g : X \to Y$ with $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$. Because Y is Hausdorff, we can chose disjoint opens $U, V \subseteq Y$ with $f(x) \in U$ and $g(x) \in V$. Writing $B(K, U) = \{h : X \to Y \mid h(K) \subseteq O\}$ for $K \subset X$ and $O \subset Y$, the subsets $B(\{x\}, U)$ and $B(\{x\}, V)$ are disjoint opens of $Map(X, Y)$ that contain f and g respectively.

2. Define a metric on $Map(X, Y)$ by $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. This is well defined because X is compact. Indeed the continuous function $d(f, g) : X \to \mathbb{R}$, $x \mapsto d(f(x), g(x))$ has compact image, hence the supremum $\sup_{x \in X} d(f(x), g(x))$ is reached by some $x \in X$ (it is a maximum). Then show that the metric topology and compact open topology on $Map(X, Y)$ are both finer than each other. Proof in Hatcher Prop. A.13.

□

◊ **Exercice 3.** $Map(-, -)$ **is a bifunctor on locally compact Hausdorff spaces.**
Let X, Y, Z be locally compact Hausdorff spaces.

1. Show that the composition operation

$$- \circ - : Map(X, Y) \times Map(Y, Z) \to Map(X, Z)$$

is continuous.

2. Given $f : X \to Y$, show that the map $- \circ f : Map(Y, Z) \to Map(X, Z)$ is continuous.

3. Given $g : Y \to Z$, show that the map $g \circ - : Map(X, Y) \to Map(X, Z)$ is continuous.

Proof. 1. Let $f \in Map(X, Y)$, $g \in Map(Y, Z)$, $K \subseteq X$ be a compact set of X and $U \subseteq Z$ be an open in Z such that

$$(g \circ f)(K) \subseteq U.$$

In other words $g \circ f \in B(K, U)$.

In order to prove that $- \circ -$ is continuous at $g \circ f$, we will find an open $W \subseteq Y$ such that:

(a) its closure $\overline{W}$ is compact,

(b) $f \in B(K, W)$ and $g \in B(\overline{W}, U)$,

(c) $B(\overline{W}, U) \circ B(K, W) \subseteq B(K, U)$.

We are going to construct the open W, using the fact that Y is locally compact[1]. To do this, we need the following observation:

For each y in $f(K)$ there is a neighborhood $y \in V_y$ such that $\overline{V}_y$ is compact and $\overline{V}_y \subseteq g^{-1}(U)$.

We observe that $g^{-1}(U)$ is a neighborhood of y. Since Y is locally compact, there is some compact neighborhood $y \in K_y \subseteq g^{-1}(U)$. Being a neighborhood, it contains some open set V_y such that

$$y \in V_y \subseteq K_y \subseteq g^{-1}(U)$$

[1]We use the following definition of locally compact: For every point $y \in Y$ and open neighborhood $y \in U \subseteq Y$, there is some compact neighborhood $x \in K \subseteq U \subseteq Y$.

Since Y is Hausdorff and K_y compact, the latter must be closed. This yields

$$y \in V_y \subseteq \overline{V_y} \subseteq K_y \subseteq g^{-1}(U)$$

As closed subsets of compact spaces are compact, we get that $\overline{V_y}$ is compact, which concludes the proof of the observation.

We now are able to construct the desired open subset W.
Since $f(K)$ is compact, and $\{V_x\}_{x\in f(K)}$ is an open cover to $f(K)$, there is $x_1, \cdots, x_n \in f(K)$ such that

$$f(K) \subseteq \bigcup_{i=1}^{n} V_{x_i}.$$

We then define $W := \bigcup_{i=1}^{n} V_{x_i}$, which yields $f \in B(K, W)$. Clearly $\overline{W}$ is compact and

$$\overline{W} = \bigcup_{i=1}^{n} \overline{V}_{x_i} \subseteq g^{-1}(U).$$

The other properties (b) and (c) follow by construction.

2. The map $- \circ f$ is the restriction of $(- \circ -)$ to $\{f\} \times Map(Y, Z)$, hence is continuous.

3. Similarly $g \circ -$ is the restriction of $(- \circ -)$ to $Map(X, Y) \times \{g\}$.

□

Exercice 4. Compact-open topology vs. product topology.
Recall that for sets X, Y there is a bijection of sets $Hom_{\text{Set}}(X, Y) \cong \prod_{x\in X} Y = Y^X$.

1. Show that here is a homeomorphism $Map(*, X) \cong X$ for any space X. Here $*$ denotes the one point space.

2. If X is discrete and Y is any space, show that there is a homeomorphism $Map(X, Y) \cong \prod_{x\in X} Y$ (the compact-open topology coincides with the product topology).

Proof. 1. There is a bijection $\varphi : X \to Map(*, X)$ defined by $\varphi(x) = c_x$ where $c_x : * \to X$ is the constant map with $c_x(*) = x$. It suffices to show that φ and its inverse are continuous. A basic open of $Map(*, X)$ is of the form $B(*, U)$ where $U \subseteq X$ is open. But $\varphi^{-1}(B(*, U)) = U$ which is indeed open in X, so φ is continuous. For the inverse $\psi = \varphi^{-1}$, have $\psi^{-1}(U) = B(*, U)$ which is open in $Map(*, X)$ for any open $U \subseteq X$, so ψ is also continuous.

2. If X is discrete, there is a homeomorphism $X \cong \coprod_{x\in X} *$, so

$$Map(X, Y) \cong Map(\coprod_{x\in X} *, Y) \cong \prod_{x\in X} Map(*, Y) \cong \prod_{x\in X} Y.$$

□

◊ Exercice 5. Mapping space into a product.
Let X be a Hausdorff space.

1. Show that there is a homeomorphism $Map(X, Y \times Z) \cong Map(X, Y) \times Map(X, Z)$ for any spaces Y, Z.

2. Generalize the previous result to describe the mapping space $Map(X, \prod_i Y_i)$ for any collection of spaces $\{Y_i\}_{i \in I}$.

Proof. 1. There is already a bijection of sets given by $\varphi : f \mapsto (f_1, f_2) = (\pi_1 \circ f, \pi_2 \circ f)$. One only needs to check that it is continuous and open. Because X is Hausdorff, by a lemma seen in class subsets of the form $B(K, U \times V)$ form a sub-basis for the topology on $Map(X, Y \times Z)$ where $U \subseteq Y$ and $V \subseteq Z$ are open. One concludes by noticing that

$$\varphi^{-1}(B(K,U) \times B(L,V)) = B(K, U \times Z) \cap B(L, Y \times V);$$
$$\varphi(K, U \times V) = B(K,U) \times B(K,V);$$

for all compact $K, K' \subseteq X$ compact and open $U \subseteq Y$, $V \subseteq Z$.

2. Same strategy, but keeping in mind that an open in an infinite product $\prod_i X_i$ is of the form $\prod_i U_i$ where $U_i \subseteq X_i$ is open and all but finitely many U_i satisfy $U_i = X_i$.

□

◇ Exercice 6. Space of homeomorphisms.
Let X be a compact Hausdorff space and consider the subspace $\mathrm{Homeo}(X)$ of $Map(X, X)$ consisting of all homeomorphisms.

1. Show that composition turns $\mathrm{Homeo}(X)$ into a group.

2. Show that the inverse is a continuous map $\iota\colon \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$.

3. Show that the multiplication $m\colon \mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \xrightarrow{-\circ-} \mathrm{Homeo}(X)$ is continuous.

Proof. 1. In any category $\mathcal{C}$, you can check that the set of automorphisms $\mathrm{Aut}_{\mathcal{C}}(c) \subseteq \mathcal{C}(c, c)$ at any object $c \in \mathcal{C}$ is a group.

2. Let $f \in \mathrm{Homeo}(X)$ and $B(K, U)$ an open neighborhood of $\iota(f) = f^{-1}$ in $\mathrm{Homeo}(X)$. This means that

$$f^{-1}(K) \subseteq U \iff K \subseteq f(U) \iff f(U)^c = F(U^c) \subseteq K^c.$$

Since X is compact, the closed set U^c is compact as well, and therefore $f \in \iota^{-1}(B(K, U)) = B(U^c, K^c)$ which is open.

3. We showed in exercise 3 that $Map(X, X) \times Map(X, X) \to Map(X, X)$ is continuous. Hence the restriction to $\mathrm{Homeo}(X) \times \mathrm{Homeo}(X)$ is continuous as well.

□

Exercice 7. Free loop spaces.
Given a space X, write $\Lambda X = Map(S^1, X)$ for the free loop space of X.

1. Show that the map $X \to \Lambda X$ defined by $x \mapsto (\theta \mapsto x)$ is continuous.

2. Show that the map $S^1 \times \Lambda X \to \Lambda X$ defined by $(\theta, f) \mapsto \big(\theta' \mapsto f(\theta+\theta')\big)$ is continuous.

Proof. 1. If $B(K,U)$ is a basic open of ΛX and we denote $c : X \to \Lambda X$ the map, then $c^{-1}(B(K,U)) = U$ which is open in X by assumption, so c is continuous.

2. We give two proofs, one is categorical, while the other is a direct, hands on, use of the definitions.

- Consider the sum $+ : S^1 \times S^1 \to S^1$, $(\theta, \theta') \mapsto \theta + \theta'$, which is clearly continuous. Applying the functor $Map(-, X) : Top \to Top$ to this $+$ map, and using the enriched adjunction homeomorphism we obtain a continuous map

$$Map(S^1, X) \xrightarrow{Map(+,X)} Map(S^1 \times S^1, X) \cong Map(S^1, Map(S^1, X)),$$

given by $f \mapsto (\theta \mapsto (\theta' \mapsto f(\theta + \theta')))$. But this map is adjoint to the continuous map

$$S^1 \times Map(S^1, X) \to Map(S^1, X)$$

given by $(\theta, f) \mapsto (\theta' \mapsto f(\theta + \theta'))$, as desired.

- Let $\psi : S^1 \times \Lambda X \to \Lambda X$ the map and let $B(K,U)$ be a basic open of ΛX, together with (θ, f) such that $\psi(\theta, f) \in B(K,U)$. Then $f(\theta + K) \subseteq U$, so $f \in B(\theta + K, U)$. We can suppose K to be an interval, say $K = [s,t]$. Because K is closed and contained in the open $f^{-1}(U)$, we can find $\varepsilon > 0$ such that $[s-\varepsilon, t+\varepsilon] + \theta \subseteq f^{-1}(U)$. Now let

$$O =]\theta - \tfrac{\varepsilon}{2}, \theta + \tfrac{\varepsilon}{2}[\times B([s - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}] + \theta, U).$$

Then O is open in $S^1 \times \Lambda X$ and contains (θ, f) by the choice of ε. Moreover if $(\theta', g) \in O$, then

$$\psi(\theta', g)(K) = g(\theta' + [s,t]) \subseteq g\Big(]s - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}[+\theta\Big) \subseteq U$$

which precisely mean that $(\theta', g) \in \psi^{-1}(B(K,U))$. This shows that $f \in O \subseteq \psi^{-1}(B(K,U))$, i.e. ψ is continuous at (θ, f).

□

◊ indicates the weekly assignments.