EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 13**
Mathematics	20th May 2025

◯**Exercise 1. The Mittag-Leffler condition.**
Given a tower $\cdots \to A_2 \to A_1 \to A_0$ of abelian groups, we say that it satisfies the *Mittag-Leffler condition* if for each k, there exists $j \geq k$ such that $\operatorname{im}(A_i \to A_k) = \operatorname{im}(A_j \to A_k)$ for all $i \geq j$. We say that it satisfies the *trivial Mittag-Leffler condition* if for each k, there exist $j \geq k$ such that the map $A_j \to A_k$ is zero. In this exercise we show that if a tower $\{A_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition, then $\lim^1 A_n = 0$.

1. Show that if all the maps in the tower are surjective, then $\lim^1 A_n = 0$.
2. Show that if $\{A_n\}_n$ satisfies the trivial Mittag-Leffler condition, then $\lim^1 A_n = 0$.
3. Show that if $\{A_n\}_n$ satisfies the Mittag-Leffler condition, then $\lim^1 A_n = 0$.
   Hint: Introduce the tower $\{B_n\}$ where $B_n = \operatorname{im}(A_k \to A_n)$ for large k.

Proof. 1. Recall that $\lim^1 A_n = \operatorname{coker}(id - sh)$, so we need to show that $im(id - sh) = \prod_n A_n$. Let $(b_n) \in \prod A_n$. Define (a_n) inductively by $a_0 = 0$, and choose a_{n+1} to be such that $a_n - f_{n+1}(a_{n+1}) = b_n$ (using surjectivity). By construction $((Id-sh)(a_n))_k = a_k - f_{k+1}(a_{k+1}) = b_k$ as desired.

2. Again, we show that $(id - sh)$ is onto, so let $(b_n) \in \prod A_n$. If we work in A_n and we have some $a_m \in A_m$ for $m \geq n$, denote $\overline{a_m} := (f_{n+1} \circ f_{n+2} \circ \cdots \circ f_m)(a_m) \in A_n$ its image in the group of interest A_n. Define

$$a_n := b_n + \sum_{k>0} \overline{b_{n+k}}$$

which is well defined by the trivial Mittag-Leffler condition. But now we obtain that

$$\big((id - sh)(a_n)\big)_m = a_m - f_{m+1}(a_{m+1}) = b_m + \sum_{k>0} \overline{b_{m+k}} - f_{m+1}(b_{m+1}) - \sum_{k>0} f_{m+1}(\overline{b_{m+1+k}})$$
$$= b_m + \sum_{k>0} \overline{b_{m+k}} - \overline{b_{m+1}} - \sum_{k>0} \overline{b_{m+1+k}} = b_m,$$

as desired.

3. Define $\{B_n\}$ as hinted above. By construction the maps in the tower of $A_\bullet$ restrict to maps on $\{B_n\}$, to obtain a tower $B_\bullet$, with an inclusion of tower $B_\bullet \subseteq A_\bullet$ (compatible with the structure maps). It follows that the quotient tower $A_\bullet/B_\bullet$ is a well defined tower, so there is a short exact sequence of towers

$$0 \to B_\bullet \to A_\bullet \to (B/A)_\bullet \to 0.$$

By the lecture there is an exact sequence $\lim^1 B_\bullet \to \lim^1 A\bullet \to \lim^1(B/A)_\bullet$. But the first group is trivial by the first part, and the last group is trivial by the second part. It follows that $\lim^1 A_\bullet = 0$ as desired.

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◯**Exercise 2. Some consequences of the Whitehead theorem.**

1. Let X, Y be simply-connected CW complexes and $f : X \to Y$ be a map that induces isomorphisms on all homology groups (with integer coefficients). Show that f is a homotopy equivalence.
   <u>*Hint*</u>*: Use the relative Hurewicz theorem.*

2. Show that the weak homotopy type of a Moore space $M(A, n)$ is uniquely determined by A and n, when $n > 1$.

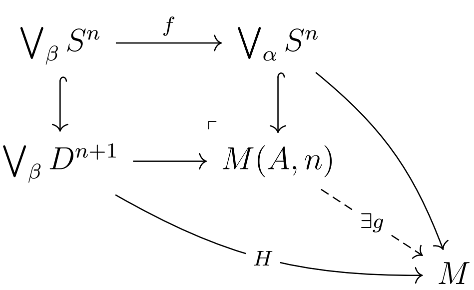
3. If $n > 1$, show that $S^n \times S^n$ and $S^n \vee S^n \vee S^{2n}$ are simply connected spaces with isomorphic homology groups, but are not homotopy equivalent. Does this fact contradict the theorem proved above?

4. What can you say about a space X that is simply connected and has the homology of a sphere S^n for some $n > 1$?

Proof.

1. As usual we can suppose that f is a cofibration (turn it into one otherwise), so that we have a good pair (Y, X) in hands. By the long exact sequence in homology, we find that $H_n(Y, X) = 0$ for all $n \geq 1$. By the long exact sequence in homotopy, we conclude that the pair is 1-connected. At this point we want to show that $\pi_2(Y, X) = 0$, and there are two ways we can go. We can use the relative Hurewicz theorem to deduce that $\pi_2(Y, X) = \pi_2(Y, X)/\pi_1(X) \cong H_2(Y, X) = 0$ (we have to mod out the action of $\pi_1(X) = 0$ since $\pi_2(Y, X)$ might not be abelian a priori), or observe by the absolute Hurewicz theorem that $\pi_2(X) \to \pi_2(Y)$ is an isomorphism. Inductively using the relative Hurewicz theorem we find that $\pi_n(Y, X) = 0$ for all n. The long exact sequence in homotopy tells us that f is a weak homotopy equivalence. By the Whitehead theorem, it is a homotopy equivalence.

2. We need to assume that a Moore space $M(A, n)$ is simply connected for $n > 1$. Otherwise the uniqueness fails, as witnessed by the "homology spheres" which have the homology of spheres, but don't have the same homotopy type. Let $M(A, n) = (\bigvee_\alpha S^n) \cup_f (\bigvee e^{n+1}_\beta)$ be the construction in the previous exercise sheet, and let M be another simply connected Moore space. By Hurewicz we find inductively that $\pi_k(M) = 0$ for $k < n$, and that $\pi_n(M) \cong H_n(M) \cong A$. Hence for each α there exists $S^n_\alpha \to M$ a generator of $\pi_n(M) = A$ such that the composition $\bigvee_\beta S^n \xrightarrow{f} \bigvee_\alpha S^n \to M$ is nullhomotopic (by H say). Hence there exists an induced map $g : M(A, n) \to M$ coming from:



Since $\pi_n(g)$ sends generators of A to generators of A, it is an isomorphism. Moreover since both spaces are $(n-1)$-connected, the Hurewicz homomorphisms for both spaces is an isomorphism. By naturality, we get a commutative square

$$\begin{array}{ccc} \pi_n(M(A,n)) & \xrightarrow{\cong} & H_n(M(A,n)) \\ \cong \big\downarrow \pi_n(g) & & \big\downarrow H_n(g) \\ \pi_n(M) & \xrightarrow{\cong} & H_n(M) \end{array}$$

which shows that $H_n(g)$ is an isomorphism. Hence g induces an isomorphism on homology groups between simply connected spaces, so it is a homotopy equivalence by the first point.

3. By the cellular approximation theorem, both spaces are simply connected. Moreover by a fast cellular homology computation, we find that they have the same homology groups, namely $\mathbb{Z}$ in dimension $2n$, and $\mathbb{Z} \oplus \mathbb{Z}$ in dimension n.

   To prove that those spaces are not homotopy equivalent, we consider their π_{2n}. First observe that the pair $((S^n \vee S^n) \times S^{2n}, (S^n \vee S^n) \vee S^{2n})$ is $(3n-1)$-connected by sheet 5, so that $\pi_{2n}(S^n \vee S^n \vee S^{2n}) \cong \pi_{2n}(S^n \vee S^n) \times \pi_{2n}(S^{2n})$ induced by the inclusion. Suppose there exists $f : S^n \times S^n \to S^n \vee S^n \vee S^{2n}$ inducing an isomorphism on π_{2n}. In particular there must exists $\alpha : S^{2n} \to S^n \times S^n$ such that the composition

$$S^{2n} \xrightarrow{\alpha} S^n \times S^n \xrightarrow{f} S^n \vee S^n \vee S^{2n} \subseteq (S^n \vee S^n) \times S^{2n} \xrightarrow{\pi_2} S^{2n} \tag{1}$$

   is homotopic to the identity. We argue that this cannot happen by showing that

   (a) the map α factors through the n-skeleton $S^n \vee S^n \subseteq S^n \times S^n$;

   (b) f sends the n-skeleton to the n-skeleton $S^n \vee S^n \subseteq S^n \vee S^n \vee S^{2n}$.

   When that's the case, the above composition (1) is the constant map, a contradiction. The second point (b) is just by the cellular approximation theorem. To show (a), let $\alpha = (u,v) \in \pi_{2n}(S^n \times S^n)$ given by maps $u, v : S^{2n} \to S^n$. We claim that α is given by the composition of the first row of the following diagram:

$$\begin{array}{ccccccc} S^{2n} & \xrightarrow{p} & S^{2n} \vee S^{2n} & \xrightarrow{u \vee v} & S^n \vee S^n & \hookrightarrow & S^n \times S^n \\ & \searrow^{1} & \big\downarrow (1,*) & & \big\downarrow (1,*) & & \big\downarrow \pi_1 \\ & & S^{2n} & \xrightarrow[u]{} & S^n & \xrightarrow[1]{} & S^n \end{array}$$

   To prove it, we must show that when projecting the composition to the first factor (respectively the second factor) of $S^n \times S^n$, we obtain $u : S^{2n} \to S^n$ (respectively $v : S^{2n} \to S^n$). But that's exactly what the rest of the commutative diagram shows! This precisely tells us that α factors through the n-skeleton $S^n \vee S^n \subseteq S^n \times S^n$ and conclude the proof.

   It does not contradict the above theorem, since we need an actual map inducing isomorphisms on homology groups to prove that the spaces are equivalent.

4. By definition it is a Moore space $M(\mathbb{Z}, n)$, so by the first point it is equivalent to the sphere S^n.

$\square$

Exercise 3. A noncontractible acyclic space.
The goal of this exercise is to construct a space X that has trivial homology groups (acyclic space) but is not contractible. This construction is due to Berrick and Casacuberta. Consider the following diagram of free groups where the index indicates the number of generators:

$$F_1 \to F_2 \to F_4 \to \cdots \to F_{2^n} \to F_{2^{n+1}} \to \cdots$$

The homomorphism $F_{2^n} \to F_{2^{n+1}}$ sends each generator x_i of F_{2^n} to the commutator $[x_{2i-1}, x_{2i}]$ in $F_{2^{n+1}}$. Denote by $P = \mathrm{colim}_n F_{2^n}$ the colimit of the tower.

1. Describe P and show that $P = [P, P]$ is a perfect group (equal to its commutator subgroup).
2. Realise this diagram as the π_1 of a diagram of wedges of circles. Denote by U the homotopy colimit (telescope) of the tower.
3. Compute the homotopy groups of U and show that $U \simeq K(P, 1)$.
4. Compute the homology groups $H_n(U; \mathbb{Z})$ of U.
   Hint: *Use that U is a CW complex of dimension 2 and the Hurewicz theorem.*
5. Conclude that U is acyclic, but not contractible.

Proof.

1. Since every map in the diagram is injective, we find that P is a group with an infinite (countable) number of generators $\{x_1, x_2, \ldots\}$, subject to the relations given by the maps, i.e. $x_i = [x_{2i-1}, x_{2i}]$ for all $i \in \mathbb{N}$. By construction every generator x_i is a simple commutator, which implies that $P \subseteq [P, P]$ as desired.
2. Define the diagram $D : (\mathbb{N}, \leq) \to Top$ on objects by $n \mapsto \bigvee_{i=1}^{2^n} S^1$, and on morphisms where $D(n \leq n+1)$ is defined for all $1 \leq i \leq 2^n$ on the i-th circle by a map $S^1 \to \bigvee S^1$ which corresponds to $[x_{2i-1}, x_{2i}] \in \pi_1(\bigvee_{i=1}^{2^{n+1}} S^1) = F_{2^{n+1}}$. It has the desired property by construction.
3. Turn this diagram D in an equivalent one D', where all the maps have been turned into cofibrations. By the lecture the homotopy colimit U of the diagram D can be computed as the strict colimit of D'. But since homotopy groups commute with filtered colimit (S^n is compact so any map $S^n \to \mathrm{colim}_i D'(i)$ factors through some $D(j)$), we obtain that

$$\pi_k(U) \cong \pi_k(\mathrm{colim}_n D'(n)) \cong \mathrm{colim}_n \pi_k(D'(n)) \cong \mathrm{colim}_n \pi_k(D(n)) \cong \mathrm{colim}_n \pi_k(\bigvee_{i=1}^{2^n} S^1)$$

   where we used that $D'(n)$ is homotopy equivalent to $D(n)$ by assumption. If $k \geq 2$ or $k = 0$ we find that $\pi_k(U) = 0$, while $\pi_1(U) = \mathrm{colim}_n F_{2^n} = P$ by definition. By uniqueness of Eilenberg MacLane spaces, U equivalent to any model of $K(P, 1)$.
4. U is connected, so $\tilde{H}_0(U) = 0$. By Hurewicz, we find that $H_1(U; \mathbb{Z}) = \pi_1(U)^{ab} = P/[P, P] = 0$. For $k \geq 2$ we use that homology preserves filtered homotopy colimits so that

$$H_k(U) \cong H_k(\mathrm{hocolim}_i D(i)) \cong \mathrm{colim}_i H_k(D(i)) = \mathrm{colim}_i 0 = 0,$$

   since $D(i)$ is a wedge of circles. To explain that, consider the composition of functors that defines singular homology:

$$Top \xrightarrow{C_\bullet} Ch(Ab) \xrightarrow{H_k} Ab.$$

The second one preserves filtered colimits (it is a standard exercise). The first one turns filtered homotopy colimits into filtered colimits. For our case of interest, consider a diagram $X : (\mathbb{N}, \leq) \to Top$ (instead of an arbitrary filtered diagram). Up to equivalence, we can suppose it consists of cofibrations only. But now a map $\Delta^k \to \mathrm{colim}_n X_n =: X$ will factor through some $X_m \subseteq X$. This construct a map $C_k(X) \to \mathrm{colim}_n C_k(X_n)$, which is an inverse to the canonical map $\mathrm{colim}_n C_k(X_n) \to C_k(X)$, and is compatible with the boundaries. This sketches the proof that $C_\bullet(\mathrm{hocolim}_n X_n) \simeq \mathrm{colim}_n C_\bullet(X_n)$.

For $k \geq 3$, one can argue that U can be constructed as a mapping telescope of wedges of spheres, hence U is the gluing of cylinders. In particular, it is a CW-complex of dimension 2, so that $H_k(U) = 0$.

5. Combining the previous points.

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Exercise 4. A non-null map that is trivial on homotopy and homology groups.
Let $\eta : S^3 \to S^2$ be the Hopf fibration and $q : T^3 \to S^3$ be the quotient map collapsing the complement of a ball in the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ to a point (equivalently, collapsing the 2-skeleton).

Show that the composite $\eta \circ q$ induces the zero map on all homotopy groups and all reduced homology groups, but is not nullhomotopic. For this last part you can assume a.a. that it is nullhomotopic and use the fact that the Hopf map is a fibration.

Proof. For $k \geq 2$ or $k = 0$ we have that $\pi_k(T^3) = 0$, while for $k = 1$ we have that $\pi_1(S^2) = 0$. Also observe that the spheres S^2 and S^3 have non-trivial reduced homology in distinct degrees. This argues that the induced map on homotopy and reduced homology groups is trivial. Suppose however that it was nullhomotopic through a null homotopy $H : \eta \circ q \simeq cst_x$. By the homotopy lifting property, we can lift it to a homotopy $L : T^3 \times I \to S^2$:

$$\begin{array}{ccc} T^3 \times \{0\} & \xrightarrow{q} & S^3 \\ \big\downarrow & \overset{\exists L}{\nearrow} & \big\downarrow \eta \\ T^3 \times I & \xrightarrow[H]{} & S^2 \end{array}$$

such that $L_0 = q : T^3 \to S^3$, and $\eta \circ L_1 = cst_x : T^3 \to S^2 \to S^3$. This last property tells us that L_1 has its image in the fiber S^1 of η over x, i.e. that L_1 factors as $T^3 \to S^1 \subseteq S^3$. I claim that L_0 and L_1 cannot be homotopic. If they were, they would induce equal maps on H_3. However L_1 is trivial on H_3 since $H_3(S^1) = 0$, while $L_0 = q$ is non-trivial. We can see this using the long exact sequence in homology for the pair $(T^3, (T^3)^{(2)})$ of the inclusion of the 2-skeleton of T^3:

$$H_3((T^3)^{(2)}) = 0 \to H_3(T^3) \xrightarrow{q} H_3(S^3) = \mathbb{Z}.$$

This shows that $H_3(q)$ is injective, and since $H_3(T^3) \cong \mathbb{Z}$ (by cellular homology, or Künneth's formula) we conclude that $H_3(q) \neq 0$.

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◯ indicates the exercises that will be presented in class.