EPFL - Autumn semester 2022-2023	J. Scherer
Homotopy theory	**Sheet 12**
MATHEMATICS	13 May 2025

◯**Exercise 1. The Hurewicz theorem for $n = 1$.**
The goal of this exercise is to show that for any connected space X, there is an isomorphism $\pi_1(X,x)^{\mathrm{ab}} \cong H_1(X)$ for any choice of basepoint $x \in X$. Here $G^{\mathrm{ab}} = G/[G,G]$ is the abelianization of a group G, obtained by factoring the subgroup $[G,G] = \{ghg^{-1}h^{-1} \mid g,h \in G\}$ of commutators.

1. Define the Hurewicz morphism $h : \pi_1(X,x) \to H_1(X)$.
2. Show that the commutator subgroup $[\pi_1(X,x), \pi_1(X,x)]$ is contained in the kernel of h.
3. Identify the morphism h when $X = S^1$ and $X = \bigvee_{i=1}^n S^1$.
4. Show that it is enough to prove the statement in the case where X is a connected CW complex of dimension 2.
5. Prove the statement using a diagram chase.
   Hint: Use the cofiber sequence defining X, apply the functors π_1 and H_1 and relate them using the map h.

Proof.

1. Let $U : I \to S^1$ identifying the two endpoints. Define $h[\lambda] = [\lambda \circ u]$, where on the left side we have pointed homotopy classes of maps, and on the right side we have a homology class. One can easily show this is well defined, and a group homomorphism (see the lecture notes).
2. H_1 is abelian, so the commutator is contained in the kernel.
3. When $X = S^1$, the generator $[1_{S^1}]$ is mapped to a generator $[u] \in H_1(S^1)$. Hence $h : \mathbb{Z} \cong \mathbb{Z}$ is an isomorphism. When $X = \bigvee_{\alpha \in A} S^1$, $\pi_1(X) \cong F(A)$ generated by the inclusions $\iota_\alpha : S^1_\alpha \hookrightarrow X$, while $H_1(X) \cong \bigoplus_A \mathbb{Z}$, generated by $[\iota_\alpha \circ u]$. This shows that $h : F(A) \to F(A)^{ab} = \bigoplus_A \mathbb{Z}$ is the abelianization functor.
4. Since π_1 and H_1 are both weak homotopy invariants, one can reduce to CW-complexes X. Consider $X^{(2)} \subseteq X$ the 2-skeleton. We know that $\pi_1(X^{(2)}) \cong \pi_1(X)$ and $H_1(X^{(2)}) \cong H_1(X)$. By naturality of the Hurewicz morphism h, we obtain that h is the abelianization for $X^{(2)}$ if and only if it is for X.
5. Suppose by the last point that X is a CW-complex of dimension 2, and suppose without loss of generality that it has a single cell (which we take as the base point). Hence the 1-skeleton is a wedge of circles $\bigvee_{\alpha \in A} S^1$ to which we attached 2-cells along an attaching map $\bigvee_\beta S^1 \xrightarrow{\vee f_\beta} \bigvee_\alpha S^1$. Hence we obtain a cofiber sequence $\bigvee_\beta S^1 \to \bigvee_\alpha S^1 \to X$. By the Puppe sequence, we have a cofiber sequence $\bigvee_\alpha S^1 \to X \to \Sigma \bigvee_\beta S^1 = \bigvee_\beta S^2$. Recall that H_1 turns cofiber sequence into long exact sequences, so that $H_1(\bigvee_\alpha S^1) \to H_1(X)$ is surjective. Moreover by cellular approximation, $\iota_* : \pi_1(X^{(1)}) \to \pi_1(X)$ is surjective. It follows that we

have the following commutative diagram, where the bottom row is exact:

$$\begin{array}{ccccc}
\pi_1(\bigvee_\beta S^1) & \xrightarrow{f_*} & \pi_1(X^{(1)}) & \xrightarrow{\iota_*} & \pi_1(X) \\
h \downarrow (-)^{ab} & & h_{X^{(1)}} \downarrow (-)^{ab} & & h_X \downarrow \\
H_1(\bigvee_\beta S^1) & \longrightarrow & H_1(X^{(1)}) & \longrightarrow & H_1(X)
\end{array}$$

We want to show that the kernel of h_X is included into the commutator $[\pi_1(X), \pi_1(X)]$, so let $m \in \pi_1(X)$ such that $h_X(m) = 0 \in H_1(X)$. We want to show that there exists $z \in ker(h_{X^{(1)}}) = [F(A), F(A)]$ which is sent to m. By surjectivity of ι_*, there exists a preimage $x \in \pi_1(X^{(1)})$. By diagram chasing, there exists $y \in \pi_1(\bigvee_\beta S^1)$ such that $x - f_*(y) \in [F(A), F(A)]$ which is sent to m, as desired.

□

Exercise 2. The relative Hurewicz map.
Define an analogue of the Hurewicz map for a pair (X, A), and show it is a group morphism for $n > 1$.

Proof. We want to define $H : \pi_n(X, A) \to H_n(X, A)$. Let $\alpha : (D^n, S^{n-1}) \to (X, A)$. Since $H_n(D^n, S^{n-1}) \cong H_n(D^n/S^{n-1}) \cong H_n(S^n) \cong \mathbb{Z}$, we can employ the same technique for the non relative case and define $h(\alpha) = \alpha_*(u_n)$. In a same fashion, it is a group homomorphism. Recall that the addition in $\pi_n(X, A)$ is induced by the cogroup structure on (D^n, S^{n-1}), and for $[\alpha], [\beta]$, their addition is given by the class of

$$(D^n, S^{n-1}) \xrightarrow{p} (D^n, S^{n-1}) \vee (D^n, S^{n-1}) \xrightarrow{\alpha \vee \beta} (X, A) \vee (X, A) \xrightarrow{\nabla} (X, A).$$

By the additivity axiom, homology H_n turns wedges into direct sums, while p induces the diagonal and ∇ induces the addition. This exactly tells us that h is a group homomorphsim (like in the lecture note for the non relative case). □

Exercise 3. Eilenberg-MacLane spaces and cohomology.

1. Show that the space $K(\mathbb{Z}, n)$ can be constructed as a Postnikov section of S^n, and show such a space is determined by its homotopy groups up to homotopy equivalence.

2. Show that $\Omega K(\mathbb{Z}, n+1) \simeq K(\mathbb{Z}, n)$.

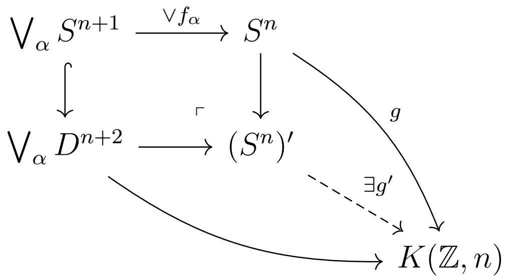
3. Show that $[-, K(\mathbb{Z}, n)]_*$ defines a (contravariant) functor $Top_*^{\mathrm{op}} \to Ab$ from the category of pointed spaces to that of abelian groups. Here $[-, -]_*$ denotes homotopy classes of pointed maps. Show that this functor sends homotopy equivalences to isomorphisms.

4. Show that a cofibration $A \hookrightarrow X \to X/A$ induces a long exact sequence of abelian groups

$$\cdots \leftarrow [A, K(\mathbb{Z}, n)]_* \leftarrow [X, K(\mathbb{Z}, n)]_* \leftarrow [X/A, K(\mathbb{Z}, n)]_* \leftarrow [A, K(\mathbb{Z}, n+1)]_* \leftarrow \cdots.$$

5. Show that there are natural isomorphisms $[X, K(\mathbb{Z}, n)]_* \cong [\Sigma X, K(\mathbb{Z}, n+1)]_*$.

Proof. 1. The first assertion is trivial, since sphere don't have lower homotopy groups and $\pi_n(S^n) = \mathbb{Z}$. For the second assertion, suppose that $K(\mathbb{Z}, n)$ is a different model. We show it is equivalent to $S^n[n]$ the n-th Postikov section of S^n. Recall the construction of $S^n[n]$ and construct a map $S^n[n] \to K(\mathbb{Z}, 1)$ inductively, which induces an isomorphism on π_n. We will not do the complete construction, but the first stage is defined like this, where f_α is a set of generators, and g is a generator:



It follows that $K(\mathbb{Z}, n)$ and $S^n[n]$ are in the same equivalence class for the relation of sheet 9.

2. Using that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$, we obtain that $\Omega K(\mathbb{Z}, n+1)$ is a $K(\mathbb{Z}, n)$. Conclude by the first point.

3. Since $K(\mathbb{Z}, n)$ is a double loop space $\Omega^2 K(\mathbb{Z}, n+2)$, it is an H-abelian group (by Eckmann-Hilton). Hence $[-, K(\mathbb{Z}, n)]_* : Top_*^{op} \to Set$ takes value in abelian group and abelian group homomorphisms.

4. We know that cofiber sequences are h-coexact. Using that $\Sigma \dashv \Omega$ and the second point, we get the result.

5. By the $\Sigma \dashv \Omega$ adjunction.

□

◯Exercise 5. Construction of a Moore space $M(A, n)$.
In this exercise we construct a Moore space of type $M(A, n)$, given an abelian group A and integer $n \geq 2$. This is a space X with the property that $H_n(X) \cong A$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$.

1. Find a short exact sequence of groups $0 \to K \xrightarrow{\varphi} F \to A \to 0$ where K, F are free abelian groups.

2. Construct a map of spaces f between two wedges of n-spheres such that $\pi_n(f) = H_n(f) = \varphi$.

3. Show that the homotopy cofiber of f is a Moore space of type $M(A, n)$.

4. Construct $K(A, n)$ as a Postnikov section of a Moore space of type $M(A, n)$.

Proof. 1. Let T be a set of generator of A, and consider a surjection $F(T) \twoheadrightarrow A$, where $F(T)$ is free abelian on T. Let $F(S) \hookrightarrow F(T)$ be the inclusion of the kernel, which is free abelian generated by some set S. Then $0 \to F(S) \to F(T) \to A \to 0$ is a short exact sequence.

2. For each generator $s \in S$, $\varphi(s) = \sum_t n_{s,t} \cdot t \in F(T)$ for some $n_{s,t} \in \mathbb{Z}$ which are almost everywhere 0. Let $N_s = \{t | n_{s,t} \neq 0\}$ be the support of $\varphi(s)$. Define a map $\bigvee_S S^n_s \to \bigvee_T S^n$ defined on the s-th component by

$$S^n_s \xrightarrow{p^{|N_s|}} \bigvee_{N_s} S^n \xrightarrow{\bigvee n_{s,t}} \bigvee_{N_s} S^n \hookrightarrow \bigvee_{t \in T} S^n,$$

where the first map is an iterated pinch map, and $n_{s,t} : S^n \to S^n$ is a map of degree $n_{s,t}$. Recall from sheet 6 that $\pi_n(\bigvee S^n) \cong \bigoplus \pi_n(S^n) \cong \bigoplus \mathbb{Z} \cong H_n(\bigvee S^n)$ and that the above maps induces $s \mapsto \varphi(s)$ by construction.

In fact we only showed the result for finite wedges in sheet 6. Let us generalize it to any wedge indexed by a set I. Note that $\bigvee_I S^n \cong \operatorname{colim}_{J \subseteq I \text{finite}} \bigvee_J S^n$. Moreover there is a canonical map $\operatorname{colim}_{J \subseteq I \text{finite}} \pi_n(\bigvee_J S^n) \to \pi_n(\operatorname{colim}_{J \subseteq I \text{finite}} \bigvee_J S^n)$. Now if I'm given $\alpha : S^n \to \bigvee_I S^n$, we know by compactness that α will touch a finitely number of spheres, indexed by a finite set I. This yields a map in the other direction $\pi_n(\operatorname{colim}_{J \subseteq I \text{ finite}} \bigvee_J S^n) \to \operatorname{colim}_{J \subseteq I \text{ finite}} \pi_n(\bigvee_J S^n)$, which is an inverse. But now using that we know the result of finite wedges we obtain

$$\begin{aligned} \pi_n(\bigvee_I S^n) &\cong \pi_n(\operatorname{colim}_{J \subseteq I \text{ finite}} \bigvee_J S^n) \cong \operatorname{colim}_{J \subseteq I \text{ finite}} \pi_n(\bigvee_J S^n) \\ &\cong \operatorname{colim}_{J \subseteq I \text{ finite}} \bigoplus_J \mathbb{Z} \cong \bigoplus_I \mathbb{Z}. \end{aligned}$$

This proves in general that homotopy groups commutes with filtered homotopy colimits. The same reasoning applies to show that homology groups commutes with filtered homotopy colimits, since the topological simplex Δ^k is compact as well (see exercise 3.4 of sheet 13).

3. Consider the homotopy cofiber X of the map above, so that we have a cofiber sequence $\bigvee_S S^n \to \bigvee_T S^n \to X$. By the long exact sequence in homology, by the previous point, we obtain a short exact sequence

$$0 \to H_{n+1}(X) \to F(S) \xrightarrow{\varphi} F(T) \to H_n(X) \to 0$$

which shows that $H_{n+1}(X) = 0$ and $H_n(X) \cong A$, while the sequence is everywhere else trivial (we use reduced homology for the end of the sequence).

4. The Moore space $M(A, n)$ has been constructed as a CW-complex with cells in dimension $> n-1$, which shows that it is $(n-1)$-connected. By Hurewicz, we obtain that $\pi_n(M(A, n)) \cong A$. It follows that $M(A, n)[n]$ is a $K(A, n)$.

□

○ indicates the exercises to be presented in class.