EPFL - Spring semester 2024-2025 | J. Scherer
Homotopy theory | **Sheet 11**
Mathematics | 6 May 2025

◯Exercise 1. Some computations of homotopy fibers.

1. Given a pointed space (X, x), turn the map $x : * \to X$ into a fibration and compute the homotopy fiber.
2. Same problem with $(*, 1) : Y \to X \times Y$.
3. Same problem with a constant map $X \to Y$.
4. Use the Hopf fibration to prove that $\Omega S^2 \simeq S^1 \times \Omega S^3$.
5. Given maps $f : X \to Y$ and $g : W \to Z$, turn $f \times g : X \times W \to Y \times Z$ into a fibration and identify the homotopy fiber.

Proof. 1. The homotopy fiber can be computed by turning $x : * \to X$ into a fibration by the path fibration $PX \to X$ where ($PX = \{\omega : I \to X | \omega(0) = x\}$), and taking the strict fiber, which is $\Omega_x X$.

2. Since the functor $- \times Y$ preserves fibrations (one can lift homotopies on each product component), $PX \times Y \to X \times Y$ is a fibration replacement of our map. Its fiber is $\Omega_x X$.

3. Our map is equal to the composition $X \to * \to Y$. Taking the homotopy fiber of the composition is the same as taking the homotopy fiber of the second map, yielding $\Omega_y Y$, and followed by the homotopy fiber with the first map, yielding $\Omega_y Y \times X$. This can be visualized in the following pasting of pullbacks:

$$\begin{array}{ccccc} X \times \Omega Y & \longrightarrow & \Omega Y & \longrightarrow & PY \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & * & \longrightarrow & Y \end{array}$$

4. Consider the Hopf fibration $S^1 \to S^3 \to S^2$. From the Puppe sequence for fibrations (Sheet 7, exercise 4) we know that the homotopy fiber of $S^1 \to S^3$ is ΩS^2. But any map $S^1 \to S^3$ is nullhomotopic, i.e. homotopic to a constant map. From the previous point we deduce that its fiber is $\Omega S^3 \times S^1$. Since homotopy fibers are unique up to equivalence, we obtain the result.

5. The product of two fibrations is a fibration, since we can lift the homotopies on each product component (we used it in the second point). It follows that the second map in $X \times W \simeq P(f) \times P(g) \to Y \times Z$ is a fibration. Its strict fiber computes the desired homotopy fiber, which is given by the product of the homotopy fibers.

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◯Exercise 2. The fibers of a fibration are homotopy equivalent.

Let $p : E \to B$ be a fibration. For each $b \in B$, denote $F_b = p^{-1}(\{b\})$ the fiber of p at b.

1. Let $u : I \to B$ be a path in B from b to b'. Use the HLP for F_b to define a map $\varphi_u : F_b \to F_{b'}$ between the corresponding fibers.
2. If $u \simeq v$ are homotopic paths in B, show that $\varphi_u \simeq \varphi_v$ are homotopic maps $F_b \to F_{b'}$. In particular show that the homotopy class of φ_u is well-defined.
3. Deduce that $F_b \simeq F_{b'}$ for any $b, b' \in B$ that are in the same path component.

Proof. 1. Consider the following homotopy lifting problem:

$$\begin{array}{ccc} F_b & \hookrightarrow & E \\ {\scriptstyle i_0}\downarrow & \overset{\exists L_u}{\nearrow} & \downarrow{\scriptstyle p} \\ F_b \times I & \xrightarrow[u \circ \pi_2]{} & B \end{array}$$

Since the square commutes, there exists a lift $L_u : F_b \times I \to E$ which, restricted to $F_b \times \{1\}$, corestricts to a map $\varphi_u : F_b \to F_{b'}$ by commutativity of the lower triangle.

2. Let $L_u, L_v : F_b \to E$ be the homotopies ending at φ_u and φ_v respectively, and let $H : I \times I \to B$ be a homotopy from $u = H(0, -)$ to $v = H(1, -)$ relative to the endpoints. In class we showed that we have a homeomorphism of pairs $(I \times I, I \times \{0\} \cong (I \times I, I \times \{0\} \cup \partial I \times I)$. Taking the product by X on both sides, we obtain a homeomorphism of pairs

$$(X \times I \times I, X \times I \times \{0\}) \cong (X \times I \times I, X \times I \times \{0\} \cup X \times \partial I \times I),$$

from which we deduce that fibrations have solutions for lifting problems on the second pair. Hence consider the following lifting problem

$$\begin{array}{ccccc} (F_b \times I \times \{0\}) \cup (F_b \times \partial I \times I) & & \longrightarrow & & E \\ \downarrow & & \overset{\exists}{\nearrow} & & \downarrow{\scriptstyle p} \\ (F_b \times I) \times I & \xrightarrow[\pi]{} & I \times I & \xrightarrow[H]{} & B \end{array}$$

where the above map is defined by to be the projection of F_b on $F_b \times I \times \{0\}$, L_u on $F_b \times \{0\} \times I$, and L_v on $F_b \times \{1\} \times I$. Those are compatible on intersections since the homotopies L_u and L_v start at the the projections of F_b. The square commutes by definition of the homotopies L_u and L_v, so there exists a solution $L : F_b \times I \times I \to E$. Let $K = L|_{F_b \times I \times \{1\}} : F_b \times I \to E$. The commutativity of the lower triangle implies that the homotopy K corestricts to $K : F_b \times I \to F_{b'}$, while the commutativity of the upper triangle implies that K is a homotopy $\varphi_u \simeq \varphi_v$ as desired.

If L_u and L'_u are two choices of homotopies defining φ_u and φ'_u, this shows that $\varphi_u \simeq \varphi'_u$, so the mapping $u \mapsto \varphi_u$ is well defined up to homotopy.

3. If b, b' are in the same path connected component, there exists a path $u : I \to B$ from b to b'. There exists an inverse path $\overline{u}$ from b' to b such that $u * \overline{u} \simeq_{\partial I} cst_b$ and $\overline{u} * u \simeq_{\partial I} cst_{b'}$ (relative to endpoints). We obtain maps $\varphi_u : F_b \to F'_b$ and $\varphi_{\overline{u}} : F_{b'} \to F_b$ which we show are homotopy inverse of each other. This follows since we can choose $\varphi_{cst_b} = Id_{F_b}$ and $\varphi_{cst_{b'}} = Id_{F_{b'}}$, and $\varphi_{u * \overline{u}} = \varphi_{\overline{u}} \circ \varphi_u$. For the last one, we consider $L : F_b \times I \to E$ defined by

$$(e_b, t) \mapsto \begin{cases} L_u(e_b, 2t) & 0 \leq t \leq \frac{1}{2}; \\ L_{\overline{u}}(L_u(e_b, 1), 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

which renders

$$\begin{array}{ccc} F_b & \hookrightarrow & E \\ {\scriptstyle i_0}\downarrow & \nearrow^{L} & \downarrow{\scriptstyle p} \\ F_b \times I & \xrightarrow[(u*\overline{u})\circ\pi_2]{} & B \end{array}$$

commutative, and such that $\varphi_{u*\overline{u}} := L|_{F_b\times\{1\}} = \varphi_{\overline{u}} \circ \varphi_u$, as desired.

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Exercise 3. The fundamental group and its action on the fibers.
Let $p : E \to B$ be a fibration and $b \in B$. As before denote F_b the fiber of p over $b \in B$. The interval I is given basepoint 0. Define $hAut(F_b)$ to be the set of (unpointed) homotopy classes of (unpointed) homotopy equivalences. It is the subset of $[F_b, F_b] = \pi_0 Map(F_b, F_b)$ on the maps that are homotopy equivalences.

1. Use the previous exercise to show that $hAut(F_b)$ is a group and define a group morphism $\pi_1(B, b) \to hAut(F_b)$.

2. Define an action of $\pi_1(B, b)$ on the set of path connected component $\pi_0(F_b)$.

3. Identify this action when p is the path fibration $Map_*(I, B) \to B$.

Proof. 1. It is clearly a group. Define the morphism by $[u]_* \mapsto [\varphi_{\overline{u}}]$ which is well defined by the previous exercise. It is a group homomorphism since $\varphi_{cst_b} \simeq Id_{F_b}$, and $\varphi_{u*v} \simeq \varphi_v \circ \varphi_u$. Note that the order of composition is reversed since we concatenate from left to right, while we compose functions from right to left, hence the introduction of the inverse $[\varphi_{\overline{u}}] = [\varphi_u^{-1}] = [\varphi_u]^{-1}$.

2. If $f \simeq g : F_b \to F_b$ are homotopic map, then $\pi_0(f) = \pi_0(g) : \pi_0(F_b) \to \pi_0(F_b)$ are equal. If $f \simeq g$ are homotopy equivalence, then the induced map on path connected components are bijections. This shows that the mapping $[f] \mapsto \pi_0(f)$ defines a map $hAut(F_b) \to Bij(\pi_0(F_b))$, which is a group homomorphism by functoriality of π_0. Hence we can compose the two homomorphisms we have in hand to obtain the desired action $\pi_1(B, b) \to hAut(F_b) \to Bij(\pi_0(F_b))$.

3. The fiber is the loop space $F_b = \Omega_b B$, so we obtain an action of $\pi_1(B, b)$ on itself since $\pi_0(\Omega B) \cong \pi_1(B, b)$. Given a loop $u : b \to b$, one can explicitly construct the lift $L_u : \Omega_b B \times I \to Map_*(I, B)$ appearing in the first point of the previous exercise by $(\lambda, t) \mapsto \lambda * u|_{[0,t]}$. This implies that $\varphi_u(\lambda) = \lambda * u$ and thus the action is $[u] \cdot [\lambda] = [\lambda * \overline{u}]$

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Exercise 4. A fiber sequence induced by a pair of composable maps.
Let $f : X \to Y$ and $g : Y \to Z$ be two composable pointed maps.

1. Show that there is an induced map between the homotopy fibers $\alpha : \mathrm{Fib}(g \circ f) \to \mathrm{Fib}(g)$.

2. Show that the homotopy fiber of α is homotopy equivalent to $\mathrm{Fib}(f)$.

Proof. 1. Consider the following pasting of homotopy pullbacks:

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow{\scriptstyle \alpha} & \lrcorner & \downarrow{\scriptstyle f} \\ Fib(g) & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ * & \longrightarrow & Z \end{array}$$

The first pullbacks gives the homotopy fiber of f by definition. By the pasting law for homotopy pullbacks, P is equivalent to the homotopy pullback of the composition $g \circ f$. This yields a map $Fib(g \circ f) \simeq P \to Fib(g)$.

2. By the pasting law for homotopy pullbacks, the homotopy fiber of α is equivalent to the homotopy fiber of f.

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Exercise 5*. The fundamental groupoid and its action on the fibers.
Given a space X the fundamental groupoid of X is the following category denoted ΠX : the objects are points $x \in X$, and morphisms $x \to y$ in ΠX are homotopy classes (with fixed endpoints) $[u]$ of paths $u : I \to X$ with $u(0) = x$ and $u(1) = y$.

1. Show that ΠX is a category in which every morphism is an isomorphism (a groupoid).
2. What is the set of endomorphisms $Hom_{\Pi X}(x, x)$ of an object x in ΠX?

Given a fibration $p : E \to B$, define a category $\mathcal{F}$ as follows: the objects are the different fibers $F_b = p^{-1}(\{b\})$ of p, and $Hom_{\mathcal{F}}(F_b, F_{b'}) = [F_b, F_b']$ is the set of homotopy classes of maps $F_b \to F_{b'}$.

3. Define composition of maps and show that $\mathcal{F}$ is a category.
4. Use the preceding exercise to define a functor $(\Pi B)^{op} \to \mathcal{F}$.

Remark. This functor is a truncated version of a very important functor in homotopy theory. Notice that from this functor alone, one can *not* recover the fibration p. The failure of (1-)category theory to be invariant under the homotopy relation (such as pullbacks/pushouts) can be repaired by working in higher ($\infty-$)category theory. In this framework, one can associate to any map $p : E \to B$ an ∞-functor $(\Pi_\infty B)^{op} \to \mathcal{F}_\infty$. From this functor one can actually recover the map $p : E \to B$, and this forms an equivalence of (∞)-groupoids $(Top/B)^{\simeq} \simeq Map((\Pi_\infty B)^{op}, \mathcal{F}_\infty)$.

Proof. 1. By construction.

2. It is $\pi_1(X, x)$.
3. By definition.
4. On objects: $b \mapsto F_b$ and on morphisms by $[u] \mapsto [\varphi_u]$.

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◯ indicates the exercises to be presented in class.