EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 10**
MATHEMATICS	29 April 2025

◯**Exercise 1. Join of topological spaces.**
Define the join of two spaces X and Y to be the quotient $X \times Y \times I/\sim$ where the equivalence relation is generated by $(x, y, 0) \sim (x', y, 0)$ and $(x, y, 1) \sim (x, y', 1)$ for $x, x' \in X$ and $y, y' \in Y$.

1. Show that $X * Y \simeq \mathrm{hocolim}(X \leftarrow X \times Y \rightarrow Y)$ where both maps collapse one wedge component. Draw the case $X = Y = S^0$.
2. Find a cofibration $X * Y \hookrightarrow CX \times CY$.
3. Show that the maps $X \to X * Y$ and $Y \to X * Y$ are nullhomotopic.
4. Discuss the case of two spheres and identify $S^n * S^m$.
5. Can you define a pointed version of the join? How does it differ from its unpointed version?

Proof.

1. Turn the projection $p_X : X \times Y \to X$ into a cofibration to get the mapping cylinder $M_{p_X} = X \times Y \times I/\sim$ where the relation is generated by $(x, y, 0) \sim (x, y', 0)$ for all $x \in X$ and $y, y' \in Y$. The strict pushout of $(M_{P_X} \leftarrow X \times Y \to Y)$ is $X \times Y \times I/\sim$ where the relation is that of the join. This computes the desired homotopy pushout. We observe by a drawing that $S^0 * S^0 = S^1$.
2. Consider the two cofibrations $X \hookrightarrow CX$ and $Y \hookrightarrow CY$ and notice that $X \times Y \hookrightarrow X \times CY$ turns p_X into a cofibration. Indeed this is in fact exactly the mapping cylinder construction:
$$X \times CY \cong X \times Y \times I/(x, y, 0) \sim (x, y', 0) \cong M_{p_X}.$$
Notice that the pushout of $X \times CY \hookleftarrow X \times Y \hookrightarrow CX \times Y$ is a reparametrized join, i.e. is homeomorphic to $X * Y$. But this pushout is precisely the pushout product of the two cofibrations we started with, hence the canonical map $X * Y \to CX \times CY$ is a cofibration (by the lectures).
3. The map is given $X \simeq X \times CY \to X * Y$, hence is $x \mapsto [(x, y, 1)]$ for any $y \in Y$. Consider the map $\iota_{y_0} : X \to X \times Y$ given by the inclusion of some arbitrary point $y_0 \in Y$ and consider the following diagram:
$$\begin{array}{ccccc} X & \xrightarrow{\iota_{y_0}} & X \times Y & \xrightarrow{p_X} & X \\ & & \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle i_X} \\ & & Y & \xrightarrow[i_Y]{} & X * Y \end{array}$$
which commutes up to homotopy (since it's a homotopy pushout). The top composition is equal to the identity, while the composition $p_Y \circ \iota_{y_0} = c_{y_0}$ is the constant map, hence factors through the point. It follows that $i_X = i_X \circ p_X \circ \iota_{y_0} \simeq i_Y \circ p_Y \circ \iota_{y_0} = i_Y \circ c_{y_0}$ which factors through the point, hence is nullhomotopic.

4. Using the next exercise, $S^n * S^m \simeq \Sigma(S^n \wedge S^m) \simeq \Sigma S^{n+m} \simeq S^{n+m+1}$. With the explicit construction, we actually have an homeomorphism. Define a map $S^n \times S^m \times I \to S^{n+m+1} \subseteq \mathbb{R}^{n+m+2}$ by $(\mathbf{x}, \mathbf{y}, t) \mapsto \sqrt{t}\mathbf{x} + \sqrt{1-t}\mathbf{y}$. This map is surjective, and injectivity fails on the edges, where it exactly identifies points in the same equivalence classes of the join construction. Hence it induces a bijection $S^n * S^m \to S^{n+m+1}$.

5. Define it the same way, using pointed constructions instead. Hence the pointed join becomes $(X \wedge Y) \rtimes I/\sim$ with the same identifications as the unpointed join.

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Exercise 2. Some practice with homotopy pushouts.
In this exercise, X, Y are well-pointed spaces.

1. Show that $\operatorname{hocolim}(X \leftarrow X \vee Y \to Y)$ is contractible.
2. If $f : X \to Y$ is a pointed map and $g : Y \to C(f)$ its homotopy cofiber, show that $C(f) \simeq \operatorname{hocolim}(* \leftarrow X \xrightarrow{f} Y)$
3. Show that $X * Y \simeq \Sigma(X \wedge Y)$.

Proof. 1. We give two proofs.

   (a) We can turn the maps into cofibrations by $X \vee Y \hookrightarrow CX \vee Y \simeq Y$ and $X \vee Y \hookrightarrow X \vee CY \simeq Y$. The homotopy pushout is equivalent to the strict pushout of $(X \vee CY \hookleftarrow X \vee Y \hookrightarrow CX \vee Y)$, which is $CX \vee CY$, a contractible space. This is visual, but you can check it using fubini's theorem (for strict pushouts).

   (b) We use Fubini's theorem for homotopy pushouts on

$$\begin{array}{ccccc} * & \longleftarrow & X & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y & \longrightarrow & * \end{array}$$

   which directly yields the result.

2. The homotopy colimit can be computed by turning $X \to *$ into a cofibration, and then taking the strict pushout. But $X \hookrightarrow CX$ is such a replacement, and by definition $C(f) = colim(CX \hookleftarrow X \to Y)$, hence they coincide.

3. This is an application of Fubini's theorem for homotopy pushouts applied to

$$\begin{array}{ccccc} * & \longleftarrow & X & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ * & \longleftarrow & X \vee Y & \hookrightarrow & X \times Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & Y & \longrightarrow & Y \end{array}$$

The homotopy colimit of this square is the homotopy pushout of the homotopy pushout of the rows, i.e. $hocolim(* \leftarrow X \wedge Y \rightarrow *) \simeq \Sigma(X \wedge Y)$, as well as the homotopy pushout of the homotopy pushout of the columns, i.e. $hocolim(* \leftarrow * \rightarrow X * Y) \simeq X * Y$, as desired. Note that we used the X, Y are well pointed spaces to deduce that $X \vee Y \hookrightarrow X \times Y$ is a cofibration.

□

Exercise 3. Examples of joins.

1. Recall that $S^0 = \partial I$ is the two point discrete space. What is $S^0 * S^0$?
2. Identify $S^n * S^m$ (up to homotopy) for any $n, m \geq 0$.
3. Show that $X * \{*\} \cong CX$ for any space X.

Proof. 1. See exercise 1.4

2. See exercise 1.4
3. By definition $X * \{*\} = X \times \{*\} \times I/\sim = X \times I/\sim$ where the relation is generated by $(x, 0) \sim (x', 0)$. This is precisely CX.

□

◯**Exercise 4. Pasting law for homotopy pushouts.**

1. Suppose given a homotopy commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

in which the left square $XYY'Y$ is a homotopy pushout. Prove that le right square $YZZ'Y'$ is a homotopy pushout if and only if the outer square $XZZ'X'$ is a homotopy pushout.

2. Show that the mapping cone (homotopy cofiber) of a nullhomotopic map $f : X \to Y$ is equivalent to $\Sigma X \vee Y$.
3. Show that $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$ for any well-pointed spaces X, Y.
4. Discuss the case of the torus (when $X = Y = S^1$) and compare the cell decomposition of the torus and the splitting of its suspension.
<u>Hint</u>: Use the pasting law for pushouts repeatedly. Start with the join from Exercise 2.

Proof. 1. By definition, the square $XYX'Y'$ is a homotopy pushout if and only if when one turns $Y \to X$ into a cofibration $X \hookrightarrow \tilde{X'} \xrightarrow{\simeq} X'$ with strict pushout $\tilde{Y'}$, there exists a map $\tilde{X'} \to Y'$, homotopic to $\tilde{X'} \to X' \to Y'$, making the composition $X \to \tilde{X'} \to Y'$ strictly equal to $X \to Y \to Y'$, and such that the comparison map $\tilde{Y'} \to Y$ is an equivalence.

By assumption, the left square is a homotopy pushout. We can depict the situation as follows:

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{X}' & \longrightarrow & \tilde{Y}' & \longrightarrow & \tilde{Z}' \\
\simeq\downarrow & \overset{\exists}{\dashrightarrow} & \simeq\downarrow & & \\
X' & \longrightarrow & Y' & \longrightarrow & Z'
\end{array}
$$

Notice that since pushouts preserve cofibrations, the vertical middle composition is the cofibration replacement of $Y \to Y'$ that we use. We know that the pasting law holds for strict pushout. Suppose that {second square, third square} = {right square, large square} as sets (unordered) to fix the terminology. Suppose that the second square is a homotopy pushout, so that the corresponding strict square is a strict pushout and there exists a second dotted arrow inducing a comparison map $f : \tilde{Z}' \to Z$ which is an equivalence. From the second dotted arrow one can construct a third dotted arrow for the third square, such that the induced comparison map $\tilde{Z}' \to Z$ is exactly f, hence an equivalence. This proves that the third square is a homotopy pushout, as desired.

2. A nullhomotopic map $X \to Y$ is homotopic to $X \to * \to Y$ where the second map picks the apex element of the nullhomotopy. Hence the homotopy cofiber of $X \to Y$ is the same as the homotopy cofiber of the composition $X \to * \to Y$ (by homotopy invariance of homotopy pushouts). But by the first point, this is the iterated homotopy pushout

$$
\begin{array}{ccccc}
X & \longrightarrow & * & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Sigma X & \longrightarrow & \Sigma X \vee Y
\end{array}
$$

Both squares are homotopy pushouts, hence so is the rectangle.

3. Consider the following diagram:

$$
\begin{array}{ccccc}
X \times Y & \longrightarrow & X & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & \Sigma(X \wedge Y) & \hookrightarrow & \Sigma X \vee \Sigma(X \wedge Y) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Sigma Y \vee \Sigma(X \wedge Y) & \longrightarrow & \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y.
\end{array}
$$

The top left square is a homotopy pushout by exercise 1 and exercise 2. Since the two maps $X \to X * Y$ and $Y \to X * Y$ are nullhomotopic, so using the previous point the top right and bottom left squares are homotopy pushouts. The bottom right square is a strict pushout. Since the hooked arrows are cofibrations (X and Y are well-pointed), it is also a homotopy pushout. But now using the pasting law for homotopy pushouts, the large square is also a homotopy pushout. By (homotopy) uniqueness of homotopy pushouts, we obtain $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$ as desired.

4. We have that $\Sigma T = \Sigma(S^1 \times S^1) \simeq \Sigma(S^1 \wedge S^1) \vee \Sigma S^1 \vee \Sigma S^1 \simeq S^3 \vee S^2 \vee S^2$. This equivalence might seem surprising since the torus is constructed by attaching a 2-cell to $S^1 \vee S^1$ with a non-trivial path given by $aba^{-1}b^{-1}$ in $\pi_1(S^1 \vee S^1)$. The above formula tells us that suspending this construction renders the attaching map trivial, removing this twisted identification of the boundary of the 2-cell, to obtain an actual copy of S^3. The reason is that the suspension of the attaching map yields the attaching map given by $S^2 = \Sigma S^1 \to \Sigma(S^1 \vee S^1) \cong \Sigma S^1 \vee \Sigma S^1 \to S^2 \vee S^2$, which lives in the abelian group $\pi_2(S^2 \vee S^2)$. Hence the twisted equation $aba^{-1}b^{-1}$ of the attaching map becomes trivial, rendering it nullhomotopic. The second point of this exercise tells us that the homotopy pushout, which construct $\Sigma(S^1 \times S^1)$ b attaching a cell, is $\Sigma S^2 \vee (S^2 \vee S^2)$, as desired.

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◯ indicates the exercises to be presented in class.