EPFL - Autumn semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 9**
Mathematics	15 April 2025

Exercise 1. A contractibility condition.

1. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_n \hookrightarrow X_{n+1}$ is nullhomotopic.
2. Show that the infinite sphere $S^\infty = \{x \in \mathbb{R}^\infty \mid \sum_i x_i^2 = 1\}$ is contractible. Here $\mathbb{R}^\infty$ is the union $\bigcup_n \mathbb{R}^n$.
3. More generally, show that $\Sigma^\infty X$ is contractible for any CW complex X. Here $\Sigma^\infty X$ denotes the union $\bigcup_n \Sigma^n X$.

◯**Exercise 2. Weak equivalence as an equivalence relation.**
Recall that a weak equivalence is a map $f : X \to Y$ that induces isomorphisms $f_* : \pi_n(X, x) \cong \pi_n(Y, f(x))$ for any $x \in X$. Let $\sim$ be the smallest equivalence relation on spaces such that if there is a weak equivalence $X \to Y$, then $X \sim Y$.

1. Show that $X \sim Y$ if and only if there is a finite zig-zag of weak equivalences $X \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_k \to Y$.
2. Show that $X \sim Y$ if and only if X and Y have a common CW approximation.
3. If X and Y are path-connected, is it true that $X \sim Y$ if and only if $\pi_n(X, x) \cong \pi_n(Y, y)$ for all n and any basepoints $x \in X$, $y \in Y$?
4. If X and Y are path-connected, is it true that $X \sim Y$ if and only if $H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z})$ for all n ? You can use $\mathbb{C}P^2$ and $S^2 \vee S^4$ to try this out.

◯**Exercise 3. Uniqueness of highly connected covers.**
Let X be a path connected and pointed space. Recall that a pointed map $X\langle n\rangle \to X$ is called an n-connected cover if it induces an isomorphism on π_k for $k > n$ and $\pi_k X\langle n\rangle = 0$ for $k \leq n$. Show that two n-connected covers are weakly equivalent.

Hint. You can use our favorite model for $X\langle n\rangle$ constructed from a single point by attaching cells of dimension $\geq n+1$ and compare it to any other n-connected cover in the spirit of the comparison of CW-approximations. Conclude by checking directly that the comparison map is a weak equivalence and by the description of the equivalence relation from Exercise 2.

Exercise 4. Relating retracts and homotopy retracts.

1. Show that a map $f : X \to Y$ has a left homotopy inverse if and only if $X \hookrightarrow Cyl(f)$ has a retraction.
2. Show that f is a homotopy equivalence if and only if $X \hookrightarrow Cyl(f)$ is a deformation retract.
3. Can you find a map $f : X \to Y$ such that $X \hookrightarrow Cyl(f)$ has a retraction but is not a deformation retract ?

◯ indicates the exercises to be presented in class.