EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 8**
Mathematics	8 April 2025

Exercise 1. Different homotopy types that have the same homotopy groups.
Define $\mathbb{R}P^\infty$ as the union $\bigcup_{n\geq 0}\mathbb{R}P^n$ where the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ is induced by the inclusion $S^n \hookrightarrow S^{n+1}$ given by $(x_0,\cdots x_n)\mapsto(x_0,\cdots x_n,0)$.

Show that $\mathbb{R}P^2$ and $S^2\times\mathbb{R}P^\infty$ have the same homotopy groups, but are not homotopy equivalent.

Exercise 2. Wedge of cofibrations is a cofibration.
If $i : A \to X$ and $j : B \to Y$ are cofibrations of pointed spaces, show that $i \vee j : A \vee B \to X \vee Y$ is a cofibration.

♢Exercise 3. Making the identity laws of an H-space strict.

1. If X and Y are finite CW complexes, show that $X \vee Y \to X \times Y$ is a cofibration.
   Remark : The conclusion still holds when X and Y are only assumed to be well-pointed.
2. Let (X, x) be an H-space and a pointed CW complex. Show that the multiplication $\mu : X \times X \to X$ is homotopic to a map m for which the basepoint is a strict identity.
3. Apply this strictification to ΩX (no explicit formula is expected).
4. There is another more explicit way to do this. Define the Moore loop space $\Omega_M X$ as the subspace of $map_*([0,\infty), X)\times[0,\infty[$ consisting of those pairs (α, t) where α is a path starting at x_0 and t is such that $\alpha(s) = x_0$ for all $s \geq t$, i.e. α becomes constant at x_0 after some time t. Show that $\Omega X \simeq \Omega_M X$, and define a product on $\Omega_M(X)$ such that the identity law is strict.

   Remark. The composition law you defined is also strictly associative.

♢Exercise 4. Cofibrations and the extension problem. Let $i : A \hookrightarrow X$ be a cofibration, and $f \simeq g : A \to Z$ two homotopic maps.

1. Show that f extends to a map $X \to Z$ if and only if g does.
2. Let $\varphi : Z \xrightarrow{\sim} Z'$ be a homotopy equivalence. Show that f extends to a map $X \to Z$ if and only if $\varphi\circ f$ extends to a map $X \to Z'$.
3. Let $f_0, f_1 : X \to Z$ be maps that agree on A, ie. $f_0|_A = f_1|_A$. Show that if $\varphi\circ f_0 \simeq \varphi\circ f_1$ rel. A, then $f_0 \simeq f_1$ rel. A.

Exercise 5. Cofiber sequence induced by a composable pair.
Let $f : X \to Y$ and $g : Y \to Z$ be pointed maps. Write $C(f)$ for the mapping cone of a map f.

1. Define canonical maps $\alpha : C(f) \to C(g\circ f)$ and $\beta : C(g\circ f) \to C(g)$.
2. Show that $C(f) \xrightarrow{\alpha} C(g\circ f) \xrightarrow{\beta} C(g)$ is a cofiber sequence, ie. that β is the mapping cone of the map α.

♢ indicates the weekly assignments.