EPFL - Fall semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 7**
Mathematics	1 April 2025

Exercise 1. A map homotopic to the identity.
Let X be a pointed space. Define a map $f : \Sigma X \to \Sigma X$ by the formula $\overline{(x,t)} \mapsto \overline{(x, \min(2t, 1))}$. Show that $f \simeq \mathrm{id}_{\Sigma X}$ is homotopic to the identity.

♢Exercise 2. Pushout squares preserve quotients.
Recall that an embedding is an injective map $j : A \to X$ which induces a homeomorphism $A \cong j(A)$ onto its image. Suppose the left square in the following diagram is a pushout with j an embedding.

$$\begin{array}{ccccc} A & \xrightarrow{j} & X & \xrightarrow{p} & X/A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \overline{F}} \\ B & \xrightarrow{J} & Y & \xrightarrow{q} & Y/B \end{array}$$

1. Show that J is also an embedding.
2. Define the induced map $\overline{F} : X/A \to B/Y$ on the quotients.
3. Show that $\overline{F}$ is a homeomorphism.

Remark. If we work with CW-complexes, the cofibers X/A and Y/B are called homotopy cofibers of j and J respectively, and the pushout is a homotopy pushout. We will see that the converse is also true : the left square is a homotopy pushout if and only if the homotopy cofibers are equivalent ! The same is true for homotopy pullback and homotopy fibers.

♢Exercise 3. The h-coaction on the mapping cone.
Let $f : X \to Y$ be a map of pointed spaces. Write $C(f)$ for the mapping cone of f defined as $CX \cup_f Y$ where $CX = X \wedge I$ where we take 0 to be the basepoint of I. Define a map $\mu : C(f) \to \Sigma X \vee C(f)$ by

$$\overline{(x,t)} \mapsto \begin{cases} (\,\overline{(x,2t)}\,, *) & \text{if } 0 \le t \le \frac{1}{2} \\ (*, \overline{(x, 2t-1)}\,) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

and $y \mapsto y$.

1. Show that for any pointed space Z, the map μ induces a left action of the group $[\Sigma X, Z]_*$ on the pointed set $[C(f), Z]_*$. Here $[-,-]_* = \pi_0 Map_*(-,-)$ denotes homotopy classes of pointed maps.
2. When f is the inclusion $X \hookrightarrow CX$, show that this action can be identified with left multiplication on the group $[\Sigma X, Z]_*$.

♢Exercise 4. The fiber sequence of a map.
(Only questions (1), (2), (3) are part of the assignment)
Let $f : X \to Y$ be a map of pointed spaces and consider the sequence of iterated mapping fibers (homotopy fibers)

$$\cdots \longrightarrow F(f_2) \xrightarrow{f_3} F(f_1) \xrightarrow{f_2} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y.$$

Here $f_1 : F(f) \to X$ is the mapping (homotopy) fiber of f, $f_2 : F(f_1) \to F(f)$ is the mapping (homotopy) fiber of f_1, and so on.

1. $\Diamond$ Describe the elements of $F(f_1)$ and its topology, together with the map $f_2 : F(f_1) \to F(f)$.
2. $\Diamond$ Define maps $\phi_Y : \Omega Y \to F(f_1)$ and $\psi_Y : F(f_1) \to \Omega Y$ that form a pointed homotopy equivalence. Write $j := f_2 \cdot \phi_Y : \Omega Y \to F(f)$.
3. $\Diamond$ Deduce that $F(f_2) \simeq \Omega X$ are homotopy equivalent and show that the following diagram is pointed homotopy commutative

$$\begin{array}{ccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y \\ {\scriptstyle \phi_X} \downarrow \simeq & & \simeq \uparrow {\scriptstyle \psi_Y} \\ F(f_2) & \xrightarrow[f_3]{} & F(f_1) \end{array}$$

where $-\Omega f = \iota \cdot \Omega f$ and $\iota : \Omega X \to \Omega X$ is the inversion map $\omega \mapsto \overline{\omega}$.

4. Deduce that the sequence

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{j} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$$

is h-exact.

5. Deduce that the following sequence is h-exact

$$\cdots \longrightarrow \Omega^2 F(f) \xrightarrow{\Omega^2 f_1} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega j} \Omega F(f) \xrightarrow{-\Omega f_1} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{j} F(f) \xrightarrow{f_1} X \xrightarrow{f} Y.$$

We know make some observations that will help you have new insights on this exercise, once we have developed the theory of fibrations and homotopy pullbacks.

6. Show that the fiber $f_1^{-1}(x_0)$ is homeomorphic to ΩY, and that the inclusion $\Omega Y \subset F(f)$ is the map j (hence j is an embedding). This tells us that the strict fiber of f_1 is also the homotopy fiber.
7. Likewise, show that the fiber $f_2^{-1}((x_0, c_{y_0}))$ is homeomorphic to ΩX and that the inclusion $\Omega Y \subset F(f_1)$ is $f_3 \cdot \phi_X$. However notice that ΩX is definitely not the fiber of $j : \Omega Y \to F(f)$ (it is its *homotopy* fiber!).

Exercise 5*. Relating fiber and cofiber sequences.

Let $f : X \to Y$ be a map of pointed spaces. Define a map $\zeta : F(f) \to \Omega C(f)$ by the formula

$$\zeta(x, \gamma)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \overline{(x, 2t-1)} & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

Define also $\xi : \Sigma F(f) \to C(f)$ as the adjoint of ζ. That is, $\xi(x, \gamma, t) = \zeta(x, \gamma)(t)$. Recall that the adjunction $\Sigma \dashv \Omega$ provides maps $\eta_X : X \to \Omega\Sigma X$ and $\varepsilon_X : \Sigma\Omega X \to X$ natural in X.

In the following diagram, the top row is obtained from the fiber sequence of f by application of the functor Σ, and the bottom row is obtained by applying Ω to the cofiber sequence of f.

$$\begin{array}{ccccccccccccc} & & \Sigma\Omega F(f) & \xrightarrow{\Sigma\Omega p} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma\alpha} & \Sigma F(f) & \xrightarrow{\Sigma p} & \Sigma X \\ & & \downarrow {\scriptstyle \varepsilon_{F(f)}} & & \downarrow {\scriptstyle \varepsilon_X} & & \downarrow {\scriptstyle \varepsilon_Y} & & \downarrow {\scriptstyle \xi} & & \| \\ \Omega Y & \xrightarrow{\alpha} & F(f) & \xrightarrow{p} & X & \xrightarrow{f} & Y & \xrightarrow{i} & C(f) & \xrightarrow{\pi} & \Sigma X \\ \| & & \downarrow {\scriptstyle \zeta} & & \downarrow {\scriptstyle \eta_X} & & \downarrow {\scriptstyle \eta_Y} & & \downarrow {\scriptstyle \eta_{C(f)}} & & \\ \Omega Y & \xrightarrow[\Omega i]{} & \Omega C(f) & \xrightarrow[\Omega\pi]{} & \Omega\Sigma X & \xrightarrow[\Omega\Sigma f]{} & \Omega\Sigma Y & \xrightarrow[\Omega\Sigma i]{} & \Omega\Sigma C(f) & & \end{array}$$

Show that the diagram is homotopy commutative. Which of the squares commute strictly ?
<u>Hint</u> : there are only two explicit homotopies to write.

$\lozenge$ indicates the weekly assignments.