EPFL - Spring semester 2024-2025 — J. Scherer

Homotopy theory — **Sheet 6**

Mathematics — 25 March 2025

Exercise 1. Examples of CW approximations.

1. Find a CW approximation to the quasi-circle (defined in sheet 3).
2. Find a CW approximation to the space $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\} \subset \mathbb{R}$.

◊Exercise 2. A space with prescribed homotopy groups.

1. Comparing $S^n \vee S^n$ with $S^n \times S^n$ compute $\pi_n(S^n \vee S^n)$ (the case $n = 1$ has to be treated separately).
2. Compute $\pi_n(\bigvee_{i=1}^n S^n)$ for any finite wedge of spheres (and an arbitrary wedge if you want !).
3. Let $M(\mathbb{Z}/p^k, n)$ be the $(n+1)$-dimensional Moore space $S^n \cup_{p^k} e^{n+1}$, where the top cell is attached via the degree p^k map on S^n. Show that $\pi_n M(\mathbb{Z}/p, n) \cong \mathbb{Z}/p$ by first showing it must be a quotient of $\pi_n S^n$ and then by using cellular homology to identify the order of the generator. Can you do it for $M(\mathbb{Z}/p^k, n)$?
4. For any finitely generated abelian group A, construct a connected space X with $\pi_n X \cong A$.

◊Exercise 3. Equivalent definitions of n-connectedness.

We say that a map $f : X \to Y$ is an n-equivalence if it induces isomorphisms $f_* : \pi_k(X, x) \cong \pi_k(Y, f(x))$ for $k < n$ and a surjection $\pi_n(X, x) \twoheadrightarrow \pi_n(Y, f(x))$ for $k = n$, for any basepoint $x \in X$. Show that the following properties of a space X are equivalent :

1. $\pi_0 Map_*(K, (X, x)) = 0$ for any pointed CW complex K of dimension $\leq n$ and any $x \in X$.
2. $X \to *$ is an $(n+1)$-equivalence.
3. $x : * \to X$ is an n-equivalence for any $x \in X$.
4. $\pi_k(X, x) = 0$ for all $k \leq n$ and any $x \in X$.

Hint : For $(4) \Rightarrow (1)$*, show by induction on* $m \leq n$ *that a map* $K \to X$ *factors through* $K/K^{(m)}$.

◊Exercise 4. The Whitehead tower of a space.

In this exercises all spaces are pointed.

1. Given a space X and $n \geq 0$, build an n-connected CW complex $X\langle n\rangle$ together with a map $f_n : X\langle n\rangle \to X$ that induces isomorphisms $\pi_k(X\langle n\rangle) \cong \pi_k(X)$ for all $k > n$.
   Hint : Reproduce the proof of CW approximation seen in class.
2. Modify the construction of $X\langle n\rangle$ so that for each n, there are maps $X\langle n+1\rangle \to X\langle n\rangle$ making the triangle
   $$\begin{array}{ccc} X\langle n+1\rangle & \longrightarrow & X\langle n\rangle \\ & {\scriptstyle f_{n+1}}\searrow \quad \swarrow{\scriptstyle f_n} & \\ & X & \end{array}$$
   strictly commutative
   Hint : construct $X\langle n+1\rangle$ *from* $X\langle n\rangle$.

The resulting diagram $\cdots \to X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X\langle 0\rangle \to X$ is called a Whitehead tower for X.

◊**Exercise 5. Some relations between a CW complex and its n-skeleton.**
Recall that if X and Y are CW complexes, the product $X \times Y$ has a canonical CW structure.

1. Let X be an n-connected space. Show that there exists a CW approximation $K \to X$ such that K has trivial n-skeleton, ie. $K^{(n)} = *$.
2. Let (X, x) be a pointed CW complex. Show that the inclusion $X^{(n)} \hookrightarrow X$ of the n-skeleton is an n-equivalence, ie. induces isomorphisms $\pi_k(X^{(n)}, x) \cong \pi_k(X, x)$ for $k < n$ and a surjection $\pi_n(X^{(n)}, x) \twoheadrightarrow \pi_n(X, x)$.
3. If $X \simeq Y$ are homotopy equivalent CW complexes without cells of dimension $n + 1$, show that $X^{(n)} \simeq Y^{(n)}$. Disprove the statement for arbitrary CW complexes.

◊ indicates the weekly assignments.