EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 5**
MATHEMATICS	18 March 2025

◊Exercise 1. Cellular approximation for pairs.

1. Show that the inclusion $S^{n-1} \times I \cup D^n \times 0 \subset D^n \times I$ is a stromng deformation retract.
2. Prove that the inclusion $A \hookrightarrow A \cup_f e^n = X$ verifies the HEP (Homotopy Extension Property) : Given a map $f\colon X \to Z$ and a homotopy $G\colon A \times I \to Z$ starting at $f|_A$, there exists a homotopy $F\colon X \times I \to Z$ starting at f and extending G.
3. Show that every map $f : (X, A) \to (Y, B)$ of CW pairs is homotopic through maps $(X, A) \to (Y, B)$ to a cellular map.

Exercise 2. Connectivity of some pairs. A pair (X, A) is n-connected if $\pi_k(X, A, a) = 0$ for all $k \leq n$ and any basepoint $a \in A$. For $k = 0$, we defined $\pi_0(X, A) = \pi_0(X)/i_*(\pi_0(A))$ where $i_* : \pi_0(A) \to \pi_0(X)$ is induced by the inclusion. The condition says here that A intersects every path component of X.

1. What are the connectivities of the pairs (S^n, S^k),$(\mathbb{R}P^n, \mathbb{R}P^k)$ and $(\mathbb{C}P^n, \mathbb{C}P^k)$ for $n > k$?
2. If X is a CW complex with $X^{(n)} = *$, what can you say about the connexity of ΣX and of the pair $(\Sigma X, X)$?

◊Exercise 3. Connectivity of some more pairs.

1. Show that if X and Y are CW complexes with $X^{(m)} = *$ and $Y^{(n)} = *$, then the pair $(X \times Y, X \vee Y)$ is $(m + n + 1)$-connected, as is the smash product $X \wedge Y$.
2. Prove that the inclusion $S^n \vee S^n \hookrightarrow S^n \times S^n$ induces an isomorphism on π_k for all $k < 2n - 1$ (and a surjection if $k = 2n - 1$).
3. For X a CW complex, show that the pair $(X, X^{(n)})$ is n-connected. Here $X^{(n)}$ denotes the n-skeleton of X.

◊Exercise 4. An extension criterion.

1. Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.
2. Show that a CW complex retracts onto any contractible subcomplex : given a CW pair (X, A) where A is contractible, show that there exist $r : X \to A$ with $r \circ \iota = id_A$, where $\iota : A \hookrightarrow X$ is the inclusion.

◊Exercise 5. The degree of a map $S^n \to S^n$.

The goal of this execrcise is to show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity. Thus the degree of such a map determines its homotopy class.

1. Reduce to the case where there is a point $y \in S^n$ such that $f^{-1}(y) = \{x_1, \cdots, x_k\}$ and f is an invertible linear map in the neighbourhood of each x_i.
   <u>Hint</u> : Use the lemma seen in class on PL approximation of a map $I^n \to Y$ on a polyhedron $K \subset I^n$.
2. For f as in (1), consider the composition $g \circ f$ where $g : S^n \to S^n$ collapses the complement of a small ball around y to the basepoint. Use this to reduce to the case where $k = 1$ in (1).

3. Conclude using the fact that $\mathrm{GL}_n(\mathbb{R})$ has only two path components (the proof of this fact is not required).

Exercise 6. Functoriality of cellular homology.

1. Recall the definition of cellular homology $H_*^{\mathrm{cell}}(X)$ for a CW complex X.
2. Using cellular approximation, show that a map $f : X \to Y$ between CW complexes induces well defined group morphisms $f_* : H_*^{\mathrm{cell}}(X) \to H_*^{\mathrm{cell}}(Y)$.
3. Show that for any pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of composable maps, we have $g_* \circ f_* = (g \circ f)_*$.

$\Diamond$ indicates the weekly assignments.