Exercice 1. Some examples of relative homotopy groups.

1. Compute the relative homotopy groups of the pair $(S^1 \times S^1, S^1 \vee S^1)$.
2. What can you say about the relative groups of the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$?

$\Diamond$**Exercice 2. Some properties of relative homotopy groups.**

1. If (A, a) is a pointed space, what are $\pi_n(A, A, a)$ and $\pi_n(A, a, a)$ for $n \geq 1$?
2. If $A \subset X$ is a homotopy equivalence, show that $\pi_n(X, A, a) = 0$ for $a \in A$ and $n \geq 1$.
3. If (X, A, a) is a pointed pair where X is contractible, what can you say about $\pi_n(X, A, a)$ for $n \geq 1$?
4. For a pair (X, A) of path-connected spaces and $a \in A$, show that $\pi_1(X, A, a)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, a)$ represented by loops in A based at a.
5. Show that in general, it is not possible to find a group structure on $\pi_1(X, A, a)$ such that $\pi_1(X, a) \to \pi_1(X, A, a)$ is a morphism of groups.

Exercice 3. An H-cogroup structure in pointed pairs.
Recall that the category $Top_*^{(2)}$ of pointed pairs has objects (X, A, a) where $a \in A \subset X$ and morphisms $(X, A, a) \to (Y, B, b)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$ and $f(a) = b$.

Given a pointed pair (X, A, a), show that the group structure on $\pi_n(X, A, a)$ for $n \geq 2$ is induced by an H-cogroup structure on $(D^n, S^{n-1}, *)$ in the category $Top_*^{(2)}$ of pairs of pointed spaces.

Exercice 4. Extending the sequence of a pair.

1. For a pointed pair (X, A, a), show that the sequence $\pi_1(X, a) \to \pi_1(X, A, a) \xrightarrow{\partial} \pi_0(A, a) \to \pi_0(X, a)$ is exact.
2. How can you define $\pi_0(X, A, a)$ so that $\cdots \to \pi_0(A, a) \to \pi_0(X, a) \to \pi_0(X, A, a) \to 0$ is exact ?

$\Diamond$**Exercice 5. The long exact sequence of a triple.**
A pointed triple (X, A, B, b) consists of spaces X, A, B with $B \subset A \subset X$ and a basepoint $b \in B$. For each $n \geq 1$, define a 'boundary' map $\partial : \pi_n(X, A, b) \to \pi_{n-1}(A, B, b)$ as the composite $\pi_n(X, A, b) \xrightarrow{\delta} \pi_{n-1}(A, b) \xrightarrow{i} \pi_{n-1}(A, B, b)$ where δ is the connecting map in the sequence for (X, A) and i is induced by the inclusion.

Show that there is a long exact sequence

$$\cdots \to \pi_n(A, B, b) \to \pi_n(X, B, b) \to \pi_n(X, A, b) \xrightarrow{\partial} \pi_{n-1}(A, B, b) \to \cdots \to \pi_1(X, A, b).$$

Hint. You can write down three long exact sequences of pairs and put them together in the form of a braided diagram. The rest is formal so you don't have to use the explicit formulas.

◊Exercice 6. Higher relative homotopy groups are abelian.
Let I^n be the n-cube and $J^{n-1} \subset \partial I^n = I^{n-1} \times \partial I \cup \partial I^{n-1} \times I$ be the subset $I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I$.
For triples (X, A, B) and (X', A', B'), denote $[(X, A, B), (X', A', B')]$ the set of homotopy classes of maps of triples, ie. maps $f : X \to X'$ such that $f(A) \subset A'$ and $f(B) \subset B'$.
Given a subspace $A \subset X$ and a basepoint $a \in A$, show that $\pi_n(X, A, a) \cong [(I^n, \partial I^n, J^{n-1}), (X, A, \{a\})]$.
We can hence represent elements in $\pi_n(X, A, a)$ by maps $I^n \to X$.
For $\alpha, \beta : I^n \to X$, define a map $\alpha *_i \beta : I^n \to X$ by the formula

$$(\alpha *_i \beta)(t_1, \cdots, t_n) = \begin{cases} \alpha(t_1, \cdots, 2t_i, \cdots, t_n) & \text{if } 0 \leq t_i \leq \frac{1}{2} \\ \beta(t_1, \cdots, 2t_i - 1, \cdots, t_n) & \text{if } \frac{1}{2} \leq t_i \leq 1 \end{cases}$$

1. If $n \geq 2$ and $i < n$, show that $*_i$ defines a group structure on $\pi_n(X, A, a)$.
2. If $i, j < n$ with $i \neq j$, show that $*_i$ and $*_j$ satisfy the interchange law :

$$(\alpha *_i \beta) *_j (\gamma *_i \delta) = (\alpha *_j \gamma) *_i (\beta *_j \delta).$$

3. Use the Heckmann-Hilton argument to show that $\pi_n(X, A, a)$ is an abelian group when $n \geq 3$.
4. Find an inclusion of spaces $A \subset X$ for which $\pi_2(X, A, a)$ is not abelian.

Exercice 7. The action of π_1 on π_n.
Let (X, x) be a path-connected pointed space and γ a path in X with endpoints $x, y \in X$.

1. Use γ to define an isomorphism $\varphi_\gamma : \pi_n(X, x) \cong \pi_n(X, y)$ for $n \geq 1$.
   Hint : you can represent a class in $\pi_n(X, x)$ by a map $(D^n, S^{n-1}) \to (X, a)$.
2. Restricting to loops based at x, show that $\gamma \mapsto \varphi_\gamma$ defines an action of $\pi_1(X, x)$ on $\pi_n(X, x)$.
3. Show that this action is induced from a map $S^n \to S^n \vee S^1$ by an application of the functor $[-, X]_*$.
   Hint : use the fact that $S^n \cong D^n/S^{n-1}$.
4. * Define similarly an action of $\pi_1(A, a)$ on $\pi_n(X, A, a)$ for $n \geq 1$, and show that it is induced from a map of pairs by application of the functor $[-, (X, A)]_*$.

◊ indicates the weekly assignments.