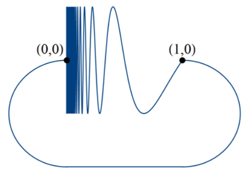
EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 3**
MATHEMATICS	4 March 2025

Exercice 1. The 'quasi-circle'.

Define the 'quasi-circle' to be a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph $y = \sin(\frac{\pi}{x})$, the segment $[-1, 1]$ in the y-axis and an arc connecting these two pieces (see picture).



Show that the quasi-circle has trivial homotopy groups, but is not contractible.

♢Exercice 2. Weak equivalences vs. homotopy equivalences

Recall that a weak equivalence is a map $f : X \to Y$ that induces isomorphisms on all homotopy groups $f_* : \pi_n(X, x) \cong \pi_n(Y, f(x))$ for all n and all base points $x \in X$. The objective of this exercise is to give an idea of how to construct a space weakly equivalent to the circle, but which is not homotopy equivalent.

1. Show that a homotopy equivalence is a weak equivalence.
2. Consider the finite space X with four points a, b, c, d whose topology is given by the following list of open subsets : $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X$. Show that X is path connected.
3. Construct a surjective (continuous) map $f : S^1 \to X$.
4. Show that the only (continuous) maps $X \to S^1$ are constant.
5. Show that the open subspace $\{a, b, c\} \subset X$ is contractible. This shows that X can be seen as a union of two contractible open subspaces whose intersection is a discrete subspace with two points.

Remark. One can prove with a Seifert-van Kampen Theorem for fundamental *groupoids* that $\pi_1 X \cong \mathbb{Z}$ and that f above induces an isomorphism on all homotopy groups. This exercise gives some plausibility to this.

Exercice 3. Higher homotopy groups are abelian.

1. Prove that the fundamental group of any H-space (not necessarily homotopy associative) is abelian.
2. Prove (again) that for any space X the homotopy groups $\pi_n(X)$ are abelian for $n \geq 2$.

♢Exercice 4. Homotopy groups and coverings.

1. Show that a covering space projection $p : E \to B$ induces isomorphisms $p_* : \pi_n(E) \cong \pi_n(B)$ for any $n \geq 2$ and any choice of basepoint $e \in E$.
2. Compute $\pi_n(S^1)$ for all $n \geq 1$.
3. Compute $\pi_n(K)$ for all $n \geq 1$, where K is the Klein bottle.
4. Compare the higher homotopy groups of $\mathbb{R}P^2$ with those of S^2.

Exercice 5. Homotopy groups of products.

1. Given a collection of path-connected spaces X_α, show that there are isomorphisms $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for any choices of basepoints $x_\alpha \in X_\alpha$.

2. Compute $\pi_n(T)$ for all $n \geq 2$, where $T = S^1 \times S^1$ is the torus (you can also use Exercise 4).

◊Exercice 6. co-H-groups.

1. Let X be a co-H-group. Show that $[X, -]_*$ defines a functor from pointed topological spaces to the category of groups.
2. Let X be a pointed space such that $[X, -]_*$ defines a functor from pointed topological spaces to the category of groups. Show that X is a co-H-group.
3. Show that a co-H-map $X \to X'$ between co-H-groups induces a group homomorphism $[X', Y]_* \to [X, Y]_*$ for any pointed space Y.

◊Exercice 7. Group objects in categories. Let $\mathcal{C}$ be a category with products and a terminal object I.

1. By analogy with the definition of H-group, define the notion of *group object* in $\mathcal{C}$ so that group objects in $Sets$ are groups.
2. Show that an object G in $\mathcal{C}$ is a group object if and only if $mor_{\mathcal{C}}(-, G) : \mathcal{C}^{op} \to Sets$ factors through the category of groups. (It is the if part which takes more work as you will have to identify a neutral element, a multiplication, and an inverse).
3. Identify all group objects in the category of groups.

Exercice 8. The category of pairs of spaces.
Write $Top_{(2)}$ for the category of pairs of spaces. Objects are pairs (X, A) where $A \subset X$ and morphisms $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$.
The set $Hom_{\mathrm{Top}_{(2)}}\big((X, A), (Y, B)\big)$ is topologized as a subspace of $Map(X, Y)$. The resulting space is denoted $Map_{(2)}\big((X, A), (Y, B)\big)$.

1. Show that $Top_{(2)}$ is indeed a category.
2. Show that the forgetful functor $Top_{(2)} \to Top$, $(X, A) \mapsto X$ has both a left and a right adjoint.

For pairs $(X, A), (Y, B)$, define $(X, A)\square(Y, B) = (X \times Y, X \times B \cup A \times Y)$.

3. Prove the exponential law for pairs of locally compact Hausdorff spaces :

$$Map_{(2)}\big(\ (X, A)\square(Y, B), (Z, C)\ \big) \cong Map_{(2)}\Big(\ (X, A), \big(\ Map_{(2)}((Y, B), (Z, C)), Map(Y, C)\ \big)\ \Big)$$

Note that there is an inclusion functor $Top_* \hookrightarrow Top_{(2)}$ given by $(X, x) \mapsto (X, \{x\})$.

4. Show that the formula $q(X, A) = (X/A, *)$ defines a functor $Top_{(2)} \to Top_*$.
5. Show that q is left adjoint to the inclusion $Top_* \hookrightarrow Top_{(2)}$. Is it an equivalence of categories ?

Exercice 9. Cylinder, cone, suspension and their reduced variants.
Given a space X, write $Cyl(X) = X \times I$, $CX = (X \times I)/X \times \{1\}$ and $SX = (X \times I)/(X \times \partial I)$ for the cylinder, cone and suspension of X respectively.
Given a pointed space X, write $\overline{Cyl}(X) = X \rtimes I$, $\overline{C}X = (X \rtimes I)/X \times \{1\}$ and $\Sigma X = (X \rtimes I)/(X \times \partial I)$ for the reduced versions of the above.

1. Show that Cyl, C, S are functors $Top \to Top$ and that $\overline{Cyl}, \overline{C}, \Sigma$ are functors $Top_* \to Top_*$.
2. Prove that there are natural transformations $Cyl \to C \to S$ and $\overline{Cyl} \to \overline{C} \to \Sigma$.

3. Prove that the following diagram of functors $Top \to Top$ is commutative

$$\begin{array}{ccccc} CylU & \longrightarrow & CU & \longrightarrow & SU \\ \downarrow & & \downarrow & & \downarrow \\ U\overline{Cyl} & \longrightarrow & U\overline{C} & \longrightarrow & U\Sigma \end{array} .$$

Here $U : Top_* \to Top$ denotes the forgetful functor.

$\Diamond$ indicates the weekly assignments.