EPFL - Spring semester 2024-2025 — J. Scherer
Homotopy theory — **Sheet 2**
Mathematics — 25 February 2025

$\Diamond$Exercice 1. Homotopies are paths in a function space.
A *homotopy* between maps $f, g\colon X \to Y$ is a map $H\colon X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. Write $H\colon f \simeq g$ in such a situation. Given a space X and $x, y \in X$, a path in X between x and y is a (continuous) map $\gamma\colon I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The set of homotopy classes between X and Y, denoted $[X, Y]$, is the set of equivalences classes of maps $X \to Y$ under the relation $\simeq$.

1. If X is locally compact Hausdorff, show that there is a bijection between the set of homotopies $f \simeq g$ and the set of paths in $Map(X, Y)$ between f and g.
2. Under the same hypotheses, show that $[X, Y] \cong \pi_0 Map(X, Y)$.

Exercice 2. Composition preserves and detects homotopy equivalences.
Let X, Y, Z be locally compact spaces.

1. If $g, g'\colon Y \to Z$ are homotopic maps, show that $g \circ -$ and $g' \circ -$ are homotopic maps $Map(X, Y) \to Map(X, Z)$.
2. Show that if $g\colon Y \to Z$ is a homotopy equivalence, then so is $g \circ -$.
3. Suppose that $g \circ -$ is a homotopy equivalence $Map(X, Y) \to Map(X, Z)$ for any space X. Show that g is a homotopy equivalence.

$\Diamond$Exercice 3. The half-smash product.
Let (X, x) be a pointed space and Y be an unpointed space. Define the half-smash product $X \rtimes Y$ to be the collapse $(X \times Y)/(\{x\} \times Y)$, with base point $\{x\} \times Y$. Define $Y_+ = Y \coprod \{*\}$ to be Y with a disjoint basepoint.

1. Prove that $X \rtimes Y \approx X \wedge Y_+$.
2. For spaces X, Y with X unpointed and Y pointed, prove the adjunction identity
$$Map_*(X_+, Y) \approx Map(X, Y).$$
3. Supppose X, Z are pointed spaces and Y is locally compact. Prove the adjunction identity
$$Map_*(X \rtimes Y, Z) \approx Map(Y, Map_*(X, Z)).$$
4. Supppose X, Y, Z are pointed spaces with Y locally compact. Prove that
$$Map_*(X \rtimes Y, Z) \approx Map_*(X, Map(Y, Z))$$
where $Map(Y, Z)$ is assumed to be pointed by the constant map $Y \to Z$ at the basepoint of Z.

Exercice 4. Some special pullbacks.

1. Let $f\colon X \to Y$ be a map and $y \in Y$. Show that the subspace $f^{-1}(y) \subseteq X$ defined by $\{x \in X \mid f(x) = y\}$ makes the following square a pullback of spaces :
$$\begin{array}{ccc} f^{-1}(y) & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \{y\} & \hookrightarrow & Y \end{array} .$$

2. Let $f\colon X \to Y$ be a map and $A \subseteq Y$. Define a subspace $B \subseteq X$ by $\{x \in X \mid f(x) \in A\}$. Find the map $g : B \to A$ that makes the following square a pullback of spaces :

$$\begin{array}{ccc} B & \hookrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \hookrightarrow & Y \end{array} .$$

(once you have identified the map g, you also have to show that the square is a pullback)

Exercice 5. Some preservation properties of adjoint functors. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ be adjoint functors $F \dashv G$. Show that F preserves pushouts squares and that G preserves pullback squares. For example for F, you have to show that if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array} .$$

is a pushout square in $\mathcal{C}$, then

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle F(h)} \\ F(C) & \xrightarrow[F(k)]{} & F(D) \end{array} .$$

is a pushout square in $\mathcal{D}$. By a similar argument you can show that a left adjoint preserves any colimit, and a right adjoint preserves any limit (such as pullbacks and products).

◊Exercice 6. The Sierpiński space.

Denote by S the Sierpiński space, defined as follows : the underlying set is $\{0, 1\}$, with open sets $\varnothing, \{1\}, \{0, 1\}$. If $A \subset X$ is an inclusion of spaces, denote by $\chi_A\colon X \to \{0, 1\}$ the *characteristic function* of A, defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

1. Given a space X, identify the set $mor_{\text{Top}}(X, S)$ of continuous maps $X \to S$.
2. For X compact Hausdorff, show that the singleton $\{\chi_X\}$ is open in $Map(X, S)$.

Denote by $\mathcal{T}_X$ the topology on X. For a subset $A \subset X$, write $\mathcal{O}_A = \{U \in \mathcal{T}_X \mid A \subset U\}$.

3. Show that $\{\mathcal{O}_K \mid K \subset X \text{ compact}\}$ is the basis for a topology on $\mathcal{T}_X$. Write $\mathcal{O}(X)$ for the topology it generates and $Open(X)$ for the space $(\mathcal{T}_X, \mathcal{O}(X))$.
4. Show that there is a homeomorphism $Map(X, S) \cong Open(X)$.

Exercice 7*. Problem with locally compact spaces.

1. Show that the category of locally compact spaces is not cocomplete.
   <u>Hint :</u> Consider for example the sequence of inclusions $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$
2. Show that a colimit of locally compact spaces is compactly generated.

◊ indicates the weekly assignments, a stared exercise (such as 7*) means it goes beyond the content of this course.