EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 13**
MATHEMATICS	20th May 2025

◯**Exercise 1. The Mittag-Leffler condition.**
Given a tower $\cdots \to A_2 \to A_1 \to A_0$ of abelian groups, we say that it satisfies the *Mittag-Leffler condition* if for each k, there exists $j \geq k$ such that $\mathrm{im}(A_i \to A_k) = \mathrm{im}(A_j \to A_k)$ for all $i \geq j$. We say that it satisfies the *trivial Mittag-Leffler condition* if for each k, there exist $j \geq k$ such that the map $A_j \to A_k$ is zero. In this exercise we show that if a tower $\{A_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition, then $\lim^1 A_n = 0$.

1. Show that if all the maps in the tower are surjective, then $\lim^1 A_n = 0$.
2. Show that if $\{A_n\}_n$ satisfies the trivial Mittag-Leffler condition, then $\lim^1 A_n = 0$.
3. Show that if $\{A_n\}_n$ satisfies the Mittag-Leffler condition, then $\lim^1 A_n = 0$.
   Hint : Introduce the tower $\{B_n\}$ where $B_n = \mathrm{im}(A_k \to A_n)$ for large k.

◯**Exercise 2. Some consequences of the Whitehead theorem.**

1. Let X, Y be simply-connected CW complexes and $f : X \to Y$ be a map that induces isomorphisms on all homology groups (with integer coefficients). Show that f is a homotopy equivalence.
   Hint : Use the relative Hurewicz theorem.
2. Show that the weak homotopy type of a Moore space $M(A, n)$ is uniquely determined by A and n, when $n > 1$.
3. If $n > 1$, show that $S^n \times S^n$ and $S^n \vee S^n \vee S^{2n}$ are simply connected spaces with isomorphic homology groups, but are not homotopy equivalent. Does this fact contradict the theorem proved above ?
4. What can you say about a space X that is simply connected and has the homology of a sphere S^n for some $n > 1$?

Exercise 3. A noncontractible acyclic space.
The goal of this exercise is to construct a space X that has trivial homology groups (acyclic space) but is not contractible. This construction is due to Berrick and Casacuberta. Consider the following diagram of free groups where the index indicates the number of generators :

$$F_1 \to F_2 \to F_4 \to \cdots \to F_{2^n} \to F_{2^{n+1}} \to \cdots$$

The homomorphism $F_{2^n} \to F_{2^{n+1}}$ sends each generator x_i of F_{2^n} to the commutator $[x_{2i-1}, x_{2i}]$ in $F_{2^{n+1}}$. Denote by $P = \mathrm{colim}_n F_{2^n}$ the colimit of the tower.

1. Describe P and show that $P = [P, P]$ is a perfect group (equal to its commutator subgroup).
2. Realise this diagram as the π_1 of a diagram of wedges of circles. Denote by U the homotopy colimit (telescope) of the tower.
3. Compute the homotopy groups of U and show that $U \simeq K(P, 1)$.
4. Compute the homology groups $H_n(U; \mathbb{Z})$ of U.
   Hint : Use that U is a CW complex of dimension 2 and the Hurewicz theorem.

5. Conclude that U is acyclic, but not contractible.

Exercise 4. A non-null map that is trivial on homotopy and homology groups.
Let $\eta : S^3 \to S^2$ be the Hopf fibration and $q : T^3 \to S^3$ be the quotient map collapsing the complement of a ball in the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ to a point (equivalently, collapsing the 2-skeleton).

Show that the composite $\eta \circ q$ induces the zero map on all homotopy groups and all reduced homology groups, but is not nullhomotopic. For this last part you can assume a.a. that it is null-homotopic and use the fact that the Hopf map is a fibration.

◯ indicates the exercises that will be presented in class.