EPFL - Spring semester 2024-2025	J. Scherer
Homotopy theory	**Sheet 12**
Mathematics	13 May 2025

◯**Exercise 1. The Hurewicz theorem for $n = 1$.**
The goal of this exercise is to show that for any connected space X, there is an isomorphism $\pi_1(X,x)^{\text{ab}} \cong H_1(X)$ for any choice of basepoint $x \in X$. Here $G^{\text{ab}} = G/[G,G]$ is the abelianization of a group G, obtained by factoring the subgroup $[G,G] = \{ghg^{-1}h^{-1} \mid g,h \in G\}$ of commutators.

1. Define the Hurewicz morphism $Hu\colon \pi_1(X,x) \to H_1(X)$.
2. Show that the commutator subgroup $[\pi_1(X,x), \pi_1(X,x)]$ is contained in the kernel of Hu.
3. Identify the morphism Hu when $X = S^1$ and $X = \bigvee_{i=1}^n S^1$.
4. Show that it is enough to prove the statement in the case where X is a connected CW complex of dimension 2.
5. Prove the statement using a diagram chase.
   __Hint__ : Use the cofiber sequence defining X, apply the functors π_1 and H_1 and relate them using the map Hu.

Exercise 2. The relative Hurewicz map.
Define an analogue of the Hurewicz map for a pair (X, A), and show it is a group homomorphism for $n > 1$.

Exercise 3. Eilenberg-MacLane spaces and cohomology.

1. Show that the space $K(\mathbb{Z}, n)$ can be constructed as a Postnikov section of S^n, and show such a space is determined by its homotopy groups up to weak homotopy equivalence.
2. Show that $\Omega K(\mathbb{Z}, n+1) \simeq K(\mathbb{Z}, n)$.
3. Show that $[-, K(\mathbb{Z}, n)]_*$ defines a (contravariant) functor $Top_*^{\text{op}} \to Ab$ from the category of pointed spaces to that of abelian groups. Here $[-,-]_*$ denotes homotopy classes of pointed maps. Show that this functor sends homotopy equivalences to isomorphisms.
4. Show that a cofibration $A \hookrightarrow X \to X/A$ induces a long exact sequence of abelian groups
$$\cdots \leftarrow [A, K(\mathbb{Z}, n)]_* \leftarrow [X, K(\mathbb{Z}, n)]_* \leftarrow [X/A, K(\mathbb{Z}, n)]_* \leftarrow [A, K(\mathbb{Z}, n+1)]_* \leftarrow \cdots .$$
5. Show that there are natural isomorphisms $[X, K(\mathbb{Z}, n)]_* \cong [\Sigma X, K(\mathbb{Z}, n+1)]_*$.

◯**Exercise 5. Construction of a Moore space $M(A, n)$.**
In this exercise we construct a Moore space of type $M(A, n)$, given an abelian group A and integer $n \geq 1$. This is a space X with the property that $H_n(X) \cong A$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$.

1. Find a short exact sequence of groups $0 \to K \xrightarrow{\varphi} F \to A \to 0$ where K, F are free abelian groups.
2. Construct a map of spaces f between two wedges of n-spheres such that $\pi_n(f) = H_n(f) = \varphi$.
3. Show that the homotopy cofiber of f is a Moore space of type $M(A, n)$.
4. Construct $K(A, n)$ as a Postnikov section of a Moore space of type $M(A, n)$.

◯ indicates the exercises to be presented in class.