◯Exercise 1. Some computations of homotopy fibers.

1. Given a pointed space (X, x), turn the map $x : * \to X$ into a fibration and compute the homotopy fiber.
2. Same problem with $(*, 1) : Y \to X \times Y$.
3. Same problem with a constant map $X \to Y$.
4. Use the Hopf fibration to prove that $\Omega S^2 \simeq S^1 \times \Omega S^3$.
5. Given maps $f : X \to Y$ and $g : W \to Z$, turn $f \times g : X \times W \to Y \times Z$ into a fibration and identify the homotopy fiber.

◯Exercise 2. The fibers of a fibration are homotopy equivalent.
Let $p : E \to B$ be a fibration. For each $b \in B$, denote $F_b = p^{-1}(\{b\})$ the fiber of p at b.

1. Let $u : I \to B$ be a path in B from b to b'. Use the HLP for F_b to define a map $\varphi_u : F_b \to F_{b'}$ between the corresponding fibers.
2. If $u \simeq v$ are homotopic paths in B, show that $\varphi_u \simeq \varphi_v$ are homotopic maps $F_b \to F_{b'}$. In particular show that the homotopy class of φ_u is well-defined.
3. Deduce that $F_b \simeq F_{b'}$ for any $b, b' \in B$ that are in the same path component.

Exercise 3. The fundamental group and its action on the fibers.
Let $p : E \to B$ be a fibration and $b \in B$. As before denote F_b the fiber of p over $b \in B$. The interval I is given basepoint 0. Define $hAut(F_b)$ to be the set of (unpointed) homotopy classes of (unpointed) homotopy equivalences. It is the subset of $[F_b, F_b] = \pi_0 Map(F_b, F_b)$ on the maps that are homotopy equivalences.

1. Use the previous exercise to show that $hAut(F_b)$ is a group and define a group morphism $\pi_1(B, b) \to hAut(F_b)$.
2. Define an action of $\pi_1(B, b)$ on the set of path connected component $\pi_0(F_b)$.
3. Identify this action when p is the path fibration $Map_*(I, B) \to B$.

Exercise 4. A fiber sequence induced by a pair of composable maps.
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two composable pointed maps.

1. Show that there is an induced map between the homotopy fibers $\alpha\colon \mathrm{Fib}(g \circ f) \to \mathrm{Fib}(g)$.
2. Show that the homotopy fiber of α is homotopy equivalent to $\mathrm{Fib}(f)$.

Exercise 5*. The fundamental groupoid and its action on the fibers.
Given a space X the fundamental groupoid of X is the following category denoted ΠX : the objects are points $x \in X$, and morphisms $x \to y$ in ΠX are homotopy classes (with fixed endpoints) $[u]$ of paths $u : I \to X$ with $u(0) = x$ and $u(1) = y$.

1. Show that ΠX is a category in which every morphism is an isomorphism (a groupoid).
2. What is the set of endomorphisms $Hom_{\Pi X}(x, x)$ of an object x in ΠX ?

Given a fibration $p : E \to B$, define a category $\mathcal{F}$ as follows : the objects are the different fibers $F_b = p^{-1}(\{b\})$ of p, and $Hom_{\mathcal{F}}(F_b, F_{b'}) = [F_b, F_b']$ is the set of homotopy classes of maps $F_b \to F_{b'}$.

3. Define composition of maps and show that $\mathcal{F}$ is a category.
4. Use the preceding exercise to define a functor $(\Pi B)^{op} \to \mathcal{F}$.

Remark. This functor is a truncated version of a very important functor in homotopy theory. Notice that from this functor alone, one can *not* recover the fibration p. The failure of (1-)category theory to be invariant under the homotopy relation (such as pullbacks/pushouts) can be repaired by working in higher ($\infty-$)category theory. In this framework, one can associate to any map $p : E \to B$ an ∞-functor $(\Pi_\infty B)^{op} \to \mathcal{F}_\infty$. From this functor one can actually recover the map $p : E \to B$, and this forms an equivalence of (∞)-groupoids $(Top/B)^{\simeq} \simeq Map((\Pi_\infty B)^{op}, \mathcal{F}_\infty)$.

○ indicates the exercises to be presented in class.