EPFL - Spring semester 2024-2025 — J. Scherer
Homotopy theory — **Sheet 10**
MATHEMATICS — 29 April 2025

◯Exercise 1. Join of topological spaces.
Define the join of two spaces X and Y to be the quotient $X \times Y \times I/\sim$ where the equivalence relation is generated by $(x, y, 0) \sim (x', y, 0)$ and $(x, y, 1) \sim (x, y', 1)$ for $x, x' \in X$ and $y, y' \in Y$.

1. Show that $X * Y \simeq \mathrm{hocolim}(X \leftarrow X \times Y \rightarrow Y)$ where both maps collapse one wedge component. Draw the case $X = Y = S^0$.
2. Find a cofibration $X * Y \hookrightarrow CX \times CY$.
3. Show that the maps $X \to X * Y$ and $Y \to X * Y$ are nullhomotopic.
4. Discuss the case of two spheres and identify $S^n * S^m$.
5. Can you define a pointed version of the join ? How does it differ from its unpointed version ?

Exercise 2. Some practice with homotopy pushouts.
In this exercise, X, Y are well-pointed spaces.

1. Show that $\mathrm{hocolim}(X \leftarrow X \vee Y \rightarrow Y)$ is contractible.
2. If $f : X \to Y$ is a pointed map and $g : Y \to C(f)$ its homotopy cofiber, show that $C(f) \simeq \mathrm{hocolim}(* \leftarrow X \xrightarrow{f} Y)$
3. Show that $X * Y \simeq \Sigma(X \wedge Y)$.

Exercise 3. Examples of joins.

1. Recall that $S^0 = \partial I$ is the two point discrete space. What is $S^0 * S^0$?
2. Identify $S^n * S^m$ (up to homotopy) for any $n, m \geq 0$.
3. Show that $X * \{*\} \cong CX$ for any space X.

◯Exercise 4. Pasting law for homotopy pushouts.

1. Suppose given a homotopy commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

   in which the left square $XYY'Y$ is a homotopy pushout. Prove that le right square $YZZ'Y'$ is a homotopy pushout if and only if the outer square $XZZ'X'$ is a homotopy pushout.
2. Show that the mapping cone (homotopy cofiber) of a nullhomotopic map $f : X \to Y$ is equivalent to $\Sigma X \vee Y$.
3. Show that $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$ for any pointed spaces X, Y.
4. Discuss the case of the torus (when $X = Y = S^1$) and compare the cell decomposition of the torus and the splitting of its suspension.
   Hint : Use the pasting law for pushouts repeatedly. Start with the join from Exercise 2.

◯ indicates the exercises to be presented in class.