We need to check the interchange law: for any $a, b, c, d : X \to Y$ consider

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "1"] & X \vee X \arrow[r, "\Delta \vee \Delta"] \arrow[dr, "\Delta_{X \vee X}"] & (X \times X) \vee (X \times X) \arrow[r, "{(a \times c) \vee (b \times d)}"] \arrow[d, phantom, "2"] & (Y \times Y) \vee (Y \times Y) \arrow[r, "m \times m"] \arrow[dr, "\nabla_{Y \times Y}"] & Y \vee Y \arrow[d, "\nabla"] \arrow[dl, phantom, "3"] \\
X \times X \arrow[r, "\psi \times \psi"'] & (X \vee X) \times (X \vee X) \arrow[rr, "(a \vee b) \times (c \vee d)"'] & & (Y \vee Y) \times (Y \vee Y) \arrow[r, "\nabla \times \nabla"'] & Y \times Y \arrow[r, "m"'] & Y
\end{tikzcd}$$

$x \mapsto$ (x, x) $\quad (b(x), d(x))$

$(x_0, x) \quad (x_0, x) \mapsto (y_0, b(x)), (y_0, d(x)) \mapsto (b(x), d(x))$

① : $\Delta(\psi(x)) = (\psi(x), \psi(x)) = (\psi \times \psi)(x, x) = (\psi \times \psi) \circ \Delta(x)$

③ commutes strictly

② We look at a map from a wedge $X \vee X$. For the second copy: ✓ □

It commutes on the nose!

Corollary 3.3

$[SX, \Omega Y]_*$ is an abelian group.

proof: SX is a coH-group, ΩY an H-group. □

§4. Relative homotopy groups

We work in Top_*. Just like H_*, there is a relative version for π_*. We work with pairs (X, A) where $a_0 = x_0 \in A \subset X$. Our main goal is to establish a long exact sequence in homotopy for pairs. We will use the pair (D^n, S^{n-1}) as model for the source, sometimes it's convenient to use $(I^n, \partial I^n)$, which is homeomorphic.

Def 4.1 The n-th relative homotopy set/group of (X, A) is $\pi_n(X, A) = [(D^n, S^{n-1}); (X, A)]_*$ for $n \geq 1$.

Homotopies of pairs are pointed and send S^{n-1} to A through the whole homotopy.

Lemma 4.2
When $n \geq 2$, the pinch map on D^n identifying the equatorial disk to a point, gives $\pi_n(X, A)$ a group structure

Remark 4.3 We have $\pi_n(X, \{x_0\}) \cong \pi_n X$. This comes from the fact that a map $(D^n, S^{n-1}) \to (X, x_0)$ sends S^{n-1} to x_0, so it corresponds to a map $D^n/S^{n-1} \approx S^n \to X$. Homotopies of pairs in this setting correspond to homotopies relative to the base point of $D^n/S^{n-1} \approx S^n$, they are pointed.

Proposition 4.4
For $n \geq 1$, $\pi_n(-,-)$ is a functor on pointed pairs of spaces. (of sets if $n = 1$, of groups if $n \geq 2$).

proof: If $f : (X, A) \longrightarrow (Y, B)$ is a map in Top_*, then it induces a morphism
$$f_* : \pi_n(X, A) \longrightarrow \pi_n(Y, B)$$
$$[\alpha] \longmapsto [f \circ \alpha]$$
$$(D^n, S^{n-1}) \xrightarrow{\alpha} (X, A) \xrightarrow{f} (Y, B).$$
$\square$

Def. 4.5 The boundary operator $\partial : \pi_n(X, A) \longrightarrow \pi_{n-1} A$ sends the homotopy class of $\alpha : (D^n, S^{n-1}) \to (X, A)$ to $[\alpha|_{S^{n-1}} : S^{n-1} \longrightarrow A]$.

Remark 4.6 If $n=1$, $D^1=[-1,1]\supset S^0=\{\pm 1\}$. Here $\partial[\alpha]$ = path connected component of $\alpha(-1)$ in A

$$\partial : \pi_1(X,A) \longrightarrow \pi_0 A$$

The boundary operator is well defined since a homotopy of pairs restricts to a homotopy of maps $S^{n-1}\longrightarrow A$. When $n\geq 2$ ∂ is a group homomorphism because the pinch map on D^n restricts to the pinch map on $S^{n-1}=\partial D^n$.

Lemma 4.7 (Compression Lemma)
A map $a:(D^n,S^{n-1})\longrightarrow (X,A)$ represents the neutral element in $\pi_n(X,A)$, for $n\geq 2$, iff it is homotopic to a map, relative to S^{n-1}, that sends D^n to A

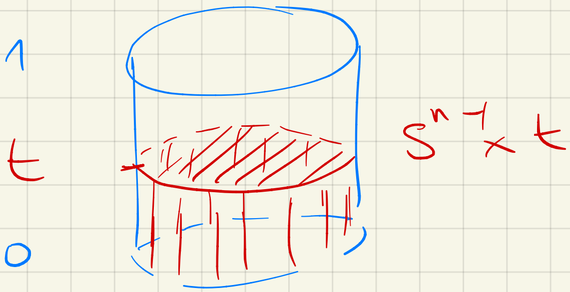
proof: If $a \underset{S^{n-1}}{\simeq} b : D^n \longrightarrow A\subset X$. Then $[a]=[b]$

Now we can contract D^n onto its base point and obtain $b\simeq c_{a_0}$ as maps of pairs $(D^n,S^{n-1})\longrightarrow (X,A)$

Conversely assume $\mathrm{Rel}[a] = [c_{x_0}]$, i.e. there exists a homotopy

$$H: D^n \times I \longrightarrow X \quad \text{s.t.} \quad H|_{S^{n-1}\times I} : S^{n-1} \times I \longrightarrow A \subset X$$

$$H|_{D^n \times 0} = a \quad \& \quad H|_{D^n\times 1} = c_{x_0}$$



Define $D_t = D^n \times t \cup S^{n-1}\times [0,t] \approx D^n$

$$\begin{array}{ccccc} D_t & & & & D^n \\ \cup & & & & \cup \\ \partial D_t & = & S^{n-1}\times 0 & \approx & S^{n-1} \end{array}$$

On D_0, $H|_{D_0} = a$, on D_1, $H|_{D_1}$ is a map $b: D^n \longrightarrow X$ whose image is inside A. The homotopy gives a relative homotopy from a to $b = H|_{D_1}$. $\square$

on $D^n \times 1$ it's constant
$S^{n-1}\times [0,1]$ it's inside A by definition of relative homotopy

Theorem 4.8 (π_* l.e.s.)

Let $i: A \subset X$ be the inclusion of a (pointed) subspace. Write $j: (X, x_0) \subset (X, A)$ for the inclusion of pairs. Then we have a l.e.s

$$\cdots \longrightarrow \pi_n A \xrightarrow{i_*} \pi_n X \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1} A \longrightarrow \cdots$$

$$\cdots \longrightarrow \underbrace{\pi_2 X}_{\text{abelian}} \xrightarrow{j_*} \underbrace{\pi_2(X, A) \xrightarrow{\partial} \pi_1 A \xrightarrow{i_*} \pi_1 X}_{\text{groups}} \xrightarrow{j_*} \underbrace{\pi_1(X, A) \longrightarrow \pi_0 A}_{\text{sets}}$$

$$\pi_0 A \longrightarrow \pi_0 X$$

proof: (1) Exactness at $\pi_n X$. First remember that $\pi_n X = \pi_n(X, x_0)$ so j_* makes sense. Then $j_* \circ i_* = (j \circ i)_*$ the map induced by $(A, x_0) \subset (X, A)$. From the Compression Lemma, the image of $(j \circ i)_*$ consists of maps of pairs whose image lies in A, hence it's trivial.

Next if $j_*[b] = 0$ where $b: S^n \longrightarrow X$ or rather $b: (D^n, S^{n-1}) \longrightarrow (X, x_0)$. By the Compression Lemma $j \circ b \simeq_{S^{n-1}} c$ whose image is in A. Then c defines a map $(D^n, S^{n-1}) \longrightarrow (A, x_0)$ and $i_*[c] = [b]$.

② Exactness at $\pi_n(X,A)$.

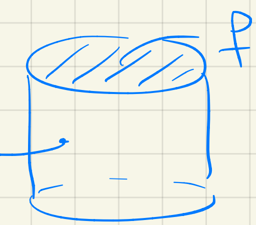
Given $b:(D^n,S^{n-1}) \to (X,x_0)$

$\partial j_*[b]$ is represented by $j\circ b|_{S^{n-1}}: S^{n-1} \to \{x_0\} \subset A$ in $\pi_{n-1}A$. It is constant, thus 0.

Next, consider $f:(D^n,S^{n-1}) \to (X,A)$ st $[f|_{S^{n-1}}]=0 \in \pi_{n-1}A$

We have a homotopy $F: S^{n-1}\times I \to A$ from f to c_{x_0}.

Define $g: D^n \cong D^n\times 1 \cup S^{n-1}\times I \xrightarrow{f\cup F} X$



Since F is entirely inside A, $f \simeq g$ as pairs $(D^n,S^{n-1}) \to (X,A)$. Since $g|_{\partial D^n} = F|_{S^{n-1}\times 0} = c_{x_0}$

So g defines a class in $\pi_n(X,x_0) = \pi_n X$. Thus

$j_*[g] = [f]$

③ Exactness at $\pi_{n-1}A$.

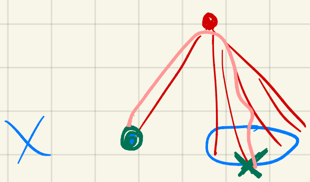
Let $f:(D^n,S^{n-1}) \to (X,A)$. Then

$i_*\partial[f] = i_*[f|_{S^{n-1}}] = [i\circ f|_{S^{n-1}}]$. This map is null-homotopic through the homotopy provided by f.

Finally if $a : S^{n-1} \to A$ s.t. $i_*[a] = 0$, i.e. $i \circ a$ is null homotopic as a map $S^{n-1} \to X$. A null homotopy can be seen as a map $f : D^n \to X$ s.t. $f|_{S^{n-1}} = a : S^{n-1} \to A$

$[f] \in \pi_n(X, A)$ s.t. $\partial [f] = [a]$. $\square$

<u>Example 4.9</u> (CX, X). Because $CX \simeq *$, $\pi_n CX = 0$ $\forall n \geq 0$. Therefore $\pi_n(CX, X) \cong \pi_{n-1} X$ for $n \geq 1$.



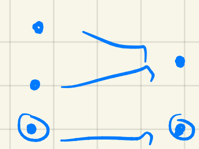
$$1 = \pi_1 CX \to \pi_1(CX, X) \xrightarrow{?} \pi_0 X \to * = \pi_0 CX$$

$$[f] \longmapsto [c_{x_0}]$$

$\pi_1(CX, X) =$ homotopy classes of maps $(D^1, S^0) \to (CX, X)$

$\bullet \; x$

homotopies send S^0 to X during the homotopy. The images of end points $\bullet$ x cannot change components, so $?$ is an iso.



III. The cellular approximation

Hatcher, Chapter 4

§ 1. CW-complexes and polyhedra

If X is a CW-complex, $X_n \subset X$ is the n-skeleton

Def 1.1 Let X, Y be CW-complexes. A map $f: X \to Y$ is *cellular* if $f(X_n) \subset Y_n$ for all $n \geq 0$

Example 1.2 If S^n is a CW-cx with two cells,
$(S^n)_0 = \ldots = (S^n)_{n-1} = *$, $S^n = (S^n)_n$.
A map $f: S^n \to S^m$ for $m > n$ is cellular iff it's constant since $(S^m)_n = *$.

Def 1.3. A *polyhedron* K in $\mathbb{R}^n$ is a subspace which is a finite union of convex compact polyhedra, ie defined by a finite number of affine inequations

$\sum a_i x_i = b$