II. Homotopy Groups

Aim:
- introduce higher homotopy groups for pointed spaces
- $\pi_n(X)$ abelian, $n \geq 2$
- (co)-H-spaces
- long exact sequence in π_* for pairs.

§1. Higher homotopy groups

(X, x_0) is a pointed space

We have already met
- $\pi_0 X = [S^0, X]_*$ where $S^0 = \{\pm 1\}$ the set of path-connected components. base point $+1$.
- $\pi_1 X = [S^1, X]_*$ the fundamental group

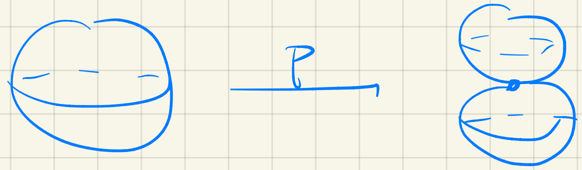
In 1932, Čech (1893–1960) introduced $\pi_n X$ (1935, Hurewicz)

Def 1.1 Let X be a pointed space, the n-th homotopy group of X is $\pi_n X = [S^n, X]_*$

The group structure comes from the pinch map $S^n \xrightarrow{\Phi} S^n \vee S^n = S^n / S^{n-1}$

equator

$n=2$:



$\nabla : X \vee X \longrightarrow X$ is the fold map $(1,1)$

Lemma 1.2

The pinch map induces a group structure on $\pi_n X$ by

$$f \star g : S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

proof: The constant map c_{x_0} is the neutral element, $\star$ is associative because

$$\begin{tikzcd}
S^n \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & S^n \vee S^n \arrow[d, "p \vee 1"] \\
S^n \vee S^n \arrow[r, "1 \vee p"'] & S^n \vee S^n \vee S^n
\end{tikzcd}$$

is commutative up to homotopy.

The inverse of a class $[f]$ is given by

$$S^n \approx S S^{n-1} \longrightarrow X.$$

$$\underset{\substack{\uparrow \\ I}}{[t}, \underset{\substack{\uparrow \\ S^{n-1}}}{s]} \longmapsto f(1-t, s)$$

$\square$

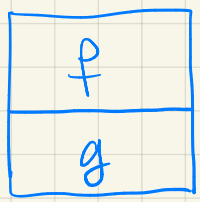
Proposition 1.3

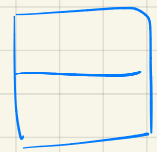
$\pi_n X$ is abelian for $n \geq 2$

proof: We construct a homotopy between $f \star g$ and $g \star f$. We do it only for $n=2$ with a drawing. Since

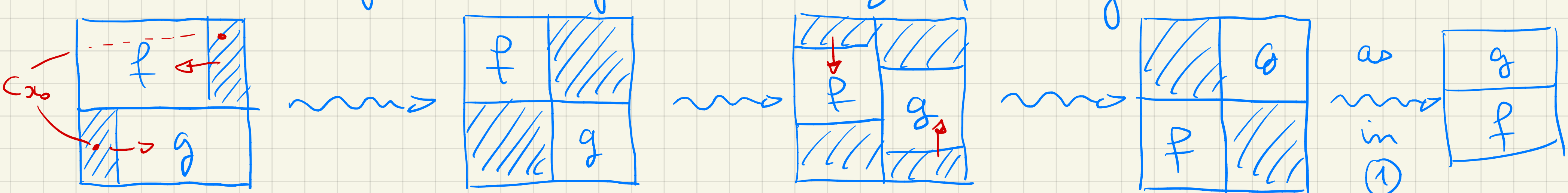
$S^2 = I \times I / \partial(I \times I)$ where the base point of S^2 is $\overline{\partial(I\times I)}$

I will draw $\boxed{f}$ for the map $f : S^2 \longrightarrow X$ and

$f \star g$ is 

where all of  is sent to $x_0 \in X$.

We start by shrinking horizontally f and g:



$0 < t < \frac{1}{3}$ (1) $\quad t = \frac{1}{3} \quad \frac{1}{3} < t < \frac{2}{3} \quad t = \frac{2}{3} \quad t = 1$

Hence $f \star g \simeq g \star f$ and $[f \star g] = [g \star f]$. $\square$

Examples 1.4

(1) $\pi_n S^n \cong \mathbb{Z} \ni 1 = [\mathrm{id}_{S^n}]$.

(2) $\pi_n S^1 = 0$ for $n > 1$ (exercise)

(3) $\pi_3 S^2 \cong \mathbb{Z} \ni 1 = [\eta]$, $\eta : S^3 \to S^2$ Hopf map

§2 H-spaces and co-H-spaces (Selick, Section 7.2)

Topologie, Série 7 : Eckmann – Hilton argument
(1917–2008) (1923–2010)

Def 2.1 An H-space or Hopf-space is a pointed space X together with a map $m : X \times X \to X$ s.t.

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "\mathrm{id}_X"'] & X \times X \arrow[d, "m"] & X \arrow[l, "i_2"'] \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd}$$

commutes up to pointed homotopy

What this says is that "$m(x, x_0) = x$" up to homotopy
"$m(x_0, x) = x$"

Def. 2.2 An H-space is *homotopy associative* if

$$\begin{tikzcd} X \times X \times X \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] \arrow[dr, phantom, "h"] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

commutes up to homotopy.

It's *homotopy commutative* if

$$(x, y) \mapsto (y, x)$$

$$\begin{tikzcd} X \times X \arrow[rr, "T"] \arrow[dr, "m"'] & \arrow[d, phantom, "h"] & X \times X \arrow[dl, "m"] \\ & X & \end{tikzcd}$$

commutes up to homotopy.

A homotopy associative H-space is an *H-group* if there is a map $i : X \longrightarrow X$ s.t.

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times i} X \times X \xrightarrow{m} X$$

is homotopic to the constant map c_{x_0}.

Examples 2.3 (1) Any topological group such as $S^0, S^1, S^3, SO(3)$ are H-groups.

(2) The unit sphere S^7 in the octonions $\mathbb{O}$ is an H-space, not homotopy associative.

(3) ΩX is an H-group for any pointed space X. m is concatenation of loops, base point is c_{x_0} and plays the role of neutral element, z is reversing the parametrization $z(\omega) = \overline{\omega}$ and $\omega \star \overline{\omega} \simeq c_{x_0}$. It's not h. commutative in general. If ΩX is h. comm, then $\pi_0 \Omega X$ becomes a commutative group $\cong \pi_1 X$, but it is not so in general.

<u>maps out of a wedge</u>

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "f"] & \\
X \arrow[r] \arrow[drr, bend right, "g"'] & X \vee Y \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dotted] & \\
 & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "id"] & \\
X \arrow[r] \arrow[drr, bend right, "id"'] & X \vee X \arrow[dr, "\nabla"] & \\
 & & X
\end{tikzcd}$$

fold

Proposition 2.5

If X is an H-group, $[W, X]_*$ is a group for any pointed space W, where the product of $f: W \to X$, $g: W \to X$ is represented by $f\circ g: W \xrightarrow{\Delta} W \times W \xrightarrow{f \times g} X \times X \xrightarrow{m} X$

proof: We have a product as defined above. The constant map c_{x_0} gives the neutral element $[c_{x_0}]$ because

$$\begin{tikzcd}
W \arrow[r, "\Delta"] & W \times W \arrow[r, "f \times c_{x_0}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright"] & X \times X \arrow[r, "m"] \arrow[dr, phantom, "\circlearrowright"] & X \\
 & W \arrow[r, "f"] & X \arrow[u, "i_1"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

commutes up to homotopy

It's associative since X is h. associative

Every class $[f]$ has an inverse represented by

$W \xrightarrow{f} X \xrightarrow{\sim} X$. $\square$

Example 2.6 When $W = S^0$, $\pi_0 X$ is a group.

Def. 2.7 A pointed space is a co-H-space X together with a comultiplication $\psi : X \longrightarrow X \vee X$ s.t.

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d] \arrow[dr, "1"] & \\ X & X \vee X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X \end{tikzcd}$$

commutes up to homotopy.

One can copy and dualize the notion of h. associative / h. commutative

and the notion of co-H-group

Proposition 2.8
If X is a co-H-group, $[X, Z]_*$ for any pointed space Z inherits a group structure, namely $[f] * [g]$ is represented by $f * g : X \xrightarrow{\psi} X \vee X \xrightarrow{f \vee g} Z \vee Z \xrightarrow{\nabla} Z$

$\mathrm{map}_*(A, B)$ is an H-group $\Rightarrow$ $\pi_0 \, \mathrm{map}_*(A, B) = [A, B]_*$ $\cong [S^0, \mathrm{map}_*(A, B)]$ is a group

Example 2.9 For any pointed space X, SX is a co-H-group

The comultiplication $SX = S^1 \wedge X \xrightarrow{p \wedge id_X} (S^1 \vee S^1) \wedge X$

by adj. $\cong$

$- \wedge X \dashv map_*(X, -)$

$(S^1 \wedge X) \vee (S^1 \wedge X) = SX \vee SX$

The inverse ν: $SX = S^1 \wedge X \xrightarrow{\nu_{S^1} \wedge X} S^1 \wedge X = SX$

Since S^1 is a co-H-group, $S^1 \wedge X$ is so as well. In particular S^n is a co-H-group.

§3 The Eckmann-Hilton trick

We study $[X, Y]_*$ for X is a co-H-group, Y an H-group.

Lemma 3.1

Let G be the underlying set of two groups $(G, \circ)$ and $(G, \star)$ sharing the same neutral element 1. Assume

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d) \text{ for all } a, b, c, d \in G$$

Then $\circ = \star$ and are abelian.

proof: $a \circ d = (a \star 1) \circ (1 \star d) = (a \circ 1) \star (1 \circ d) = a \star d$

$a \circ d = (1 \star a) \circ (d \star 1) = (1 \circ d) \star (a \circ 1) = d \star a$ □

Theorem 3.2

When X is a co-H-group, and Y is an H-group, the set $[X,Y]_*$ inherits two group structures $\circ$ and $\star$ which coincide and are abelian.

proof: We have seen that $[c_{y_0}]$ is a common unit for $\circ$ and $\star$. We need to check the interchange law: for any $a,b,c,d : X \to Y$ consider

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "1"] & X \vee X \arrow[r, "\Delta \vee \Delta"] \arrow[d, "\Delta_{X \vee X}"] \arrow[dr, phantom, "2"] & (X \times X) \vee (X \times X) \arrow[r, "{(a \times c) \vee (b \times d)}"] & (Y \times Y) \vee (Y \times Y) \arrow[r, "m \times m"] \arrow[d, "\nabla_{Y \times Y}"] \arrow[dr, phantom, "3"] & Y \vee Y \arrow[d, "\nabla"] \\
X \times X \arrow[r, "\psi \times \psi"'] & (X \vee X) \times (X \vee X) \arrow[r, "{(a \vee b) \times (c \vee d)}"'] & (Y \vee Y) \times (Y \vee Y) \arrow[r, "\nabla \times \nabla"'] & Y \times Y \arrow[r, "m"'] & Y
\end{tikzcd}$$

$(a \circ c) \star (b \circ d)$

$(a \star b) \circ (c \star d)$

We need to check the interchange law: for any $a, b, c, d : X \to Y$ consider

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "1"] & X \vee X \arrow[r, "\Delta \vee \Delta"] \arrow[d, "\Delta_{X \vee X}"] & (X \times X) \vee (X \times X) \arrow[r, "(a \times c) \vee (b \times d)"] \arrow[d, phantom, "2"] & (Y \times Y) \vee (Y \times Y) \arrow[r, "m \times m"] \arrow[d, "\nabla_{Y \times Y}"] \arrow[dr, phantom, "3"] & Y \vee Y \arrow[d, "\nabla"] \\
X \times X \arrow[r, "\psi \times \psi"'] & (X \vee X) \times (X \vee X) \arrow[r, "(a \vee b) \times (c \vee d)"'] & (Y \vee Y) \times (Y \vee Y) \arrow[r, "\nabla \times \nabla"'] & Y \times Y \arrow[r, "m"'] & Y
\end{tikzcd}$$

$x \mapsto (x, x) \mapsto (b(x), d(x))$

$(x_0, x) \mapsto (x_0, x) \mapsto (y_0, b(x)), (y_0, d(x)) \mapsto (b(x), d(x))$

① : $\Delta(\psi(x)) = (\psi(x), \psi(x)) = (\psi \times \psi)(x, x) = (\psi \times \psi) \circ \Delta(x)$

③ commutes strictly

② We look at a map from a wedge $X \vee X$. For the second copy: ✓

It commutes on the nose! $\square$