Lemma 5.3

Let $f: X \longrightarrow X$ and $Y = \mathrm{hopo}(X \xleftarrow{\nabla} X \vee X \xrightarrow{(1,f)} X)$

Then we have a cofibration sequence $\Sigma X \xrightarrow{1-f} \Sigma X \longrightarrow \Sigma Y$

Here $1-f$ indicates the operation in the co-H-group ΣX

proof. The homotopy cofiber of ∇ is $C(\nabla) = X \rtimes I / X \vee X = \Sigma X$ (reduced suspension). The Puppe sequence continues

$$X \longrightarrow \Sigma X \xrightarrow{i_1 - i_2} \Sigma(X \vee X) \simeq \Sigma X \vee \Sigma X \xrightarrow{-\Sigma\nabla} \Sigma X$$

this map is the difference of both inclusions in the wedge
Notice $-\Sigma\nabla \circ \mathrm{pinch}$ is not null-homotopic
but $-\Sigma\nabla \circ (i_1 - i_2) \simeq *$

Looking at the homotopy pushout

$$\begin{tikzcd}
X \vee X \arrow[r, "\nabla"] \arrow[d, "(1,f)"'] & X \arrow[r] \arrow[d] & \Sigma X \arrow[r, "i_1 - i_2"] \arrow[d, equal] & \Sigma X \vee \Sigma X \arrow[r, "-\Sigma\nabla"] \arrow[d, "\Sigma(1,f)"] & \Sigma X \arrow[d] \\
X \arrow[r, "g"] & Y \arrow[r] & \Sigma X \arrow[r] & \Sigma X \arrow[r, "-\Sigma g"] & \Sigma Y
\end{tikzcd}$$

The map $\Sigma X \longrightarrow \Sigma X$ is $\Sigma(1,f) \circ (i_1 - i_2) = \mathrm{id}_{\Sigma X} - \Sigma f$

$\square$

Corollary 5.4

We have a cofibration sequence $\bigvee_{n\geq 0} \Sigma X^n \xrightarrow{1-sh} \bigvee_{n\geq 0} \Sigma X^n \longrightarrow \Sigma X$

proof. We use the lemma for $f = sh : \bigvee X^n \longrightarrow \bigvee X^n$. Then

$$Y = \mathrm{hopo}\left(\bigvee X^n \xleftarrow{\nabla} (\bigvee X^n) \vee (\bigvee X^n) \xrightarrow{(1, sh)} \bigvee X^n \right)$$
$$= \mathrm{p.o}\left(\bigvee X^n \wedge I \xhookleftarrow{j} \bigvee X^n \vee \bigvee X^n \xrightarrow{(1,sh)} \bigvee X^n \right)$$
$$= \bigvee X^n \wedge I \Big/ (x_n, 1) \sim (sh(x_n), 0) \qquad x_n \in X^n, \ sh(x_n) \in X^{n+1}$$
$$= \mathrm{Tel}(X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots)$$
$$\overset{\text{homotopy invariance}}{\simeq} \mathrm{colim}(X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots) = X. \qquad \square$$

Theorem 5.5

There is an exact sequence

$$0 \to \lim{}^1 \tilde{H}^{k-1} X^n \longrightarrow \tilde{H}^k X \longrightarrow \lim \tilde{H}^k X^n \longrightarrow 0$$

So here $\tilde{H}^k X \cong \lim \tilde{H}^k X^n$

proof. The cofibration sequence from Corollary 5.4 induces a les

$$\begin{array}{ccccccccc}
\tilde H^{k+1}(\vee \Sigma X^n) & \leftarrow & \tilde H^{k+1}(\vee \Sigma X^n) & \leftarrow & \tilde H^{k+1}\Sigma X & \xleftarrow{\partial} & \tilde H^{k}(\vee \Sigma X^n) & \xleftarrow{1-sh} & \tilde H^{k}(\vee \Sigma X^n) \\
\cong & & \cong & & \cong & & \cong & & \cong \\
\prod \tilde H^k X^n & \xleftarrow{1-sh} & \prod \tilde H^k X^n & \leftarrow & \tilde H^k X & \leftarrow & \prod \tilde H^{k-1} X^n & \xleftarrow{1-sh} & \prod \tilde H^{k-1} X^n
\end{array}$$

$$\lim_n \tilde H^k X^n = \ker(1-sh) \qquad \operatorname{coker}(1-sh) = \lim^1 \tilde H^{k-1} X^n$$

<u>Note:</u> $A \xrightarrow{f} B \to C$ cofibration seq

p.o.:
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \ulcorner & \downarrow \\ CA & \to & C \end{array}$$

$$\tilde H_n A \to \tilde H_n B \oplus \underbrace{\cancel{\tilde H_n CA}}_{=0} \to \tilde H_n C \xrightarrow{\partial} \tilde H_{n-1} A$$

Finally, since X^n is the n-skeleton of X, $\tilde H^k X^n \cong \tilde H^k X^{n+1}$ for all $n > k$. Hence $\lim^1 \tilde H^{k-1} X^n = 0$. □

Theorem 5.6

$\tilde H^k(X; A) \cong [X; K(A,n)]$ for all CW-complexes X.

□

Remark 5.7 This isomorphism does not hold in general for arbitrary spaces because $[-; K(A,n)]$ is not a weak homotopy invariant functor.

The map

$$f : \mathbb{Z} \longrightarrow \{-\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \cup \{\tfrac{1}{n} \mid n \in \mathbb{N}\}$$
$$n \longmapsto \frac{1}{n} \quad \forall n \in \mathbb{Z} \setminus \{0\}$$
$$0 \longmapsto 0$$

is a weak equivalence ($\pi_n(-; x) = 0 \quad \forall n, x$). but not a homotopy equivalence.

If $\mathbb{Z}$ is our model for $K(\mathbb{Z}, 0)$, then f^* is not an iso on $[-; K(\mathbb{Z}, 0)]$.

§ 6. Postnikov theory

What is the algebraic information for a space that allows us to reconstruct the space? Homotopy groups $\pi_n X$ are not enough. For example $\mathbb{R}P^2$ and $\mathbb{R}P^2[1] \times \mathbb{R}P^2\langle 1 \rangle = \mathbb{R}P^\infty \times S^2$ are not equivalent.

We need so-called k-invariants.

Let us assume X 1-connected and look at

$p_{n+1}: X[n+1] \longrightarrow X[n]$ (Postnikov tower).

The homotopy fiber is $K(\pi_{n+1}X, n+1)$ (from the π_*-les).

Proposition 6.1

Let $K(A, n+1) \longrightarrow X \xrightarrow{p} Y$ be a fibration seq. of 1-connected spaces, then there is a map $k: Y \to K(A, n+2)$ s.t. $p: X \to Y$ is homotopy eq to $\mathrm{hoFib}(k)$.

idea: Turn p into a cofibration and think of (Y, X) as a pair. The π_*-les is

$$A \to \pi_{n+1}X \twoheadrightarrow \pi_{n+1}Y \to \underbrace{\pi_{n+1}(Y,X)}_{} \xrightarrow{\partial} \pi_n X \xrightarrow{\cong} \pi_n Y \to 0$$

Hurewicz relative: $\tilde{H}_k(Y,X) = \overset{=0}{0}$ for $k \leq n+1$

$$\tilde{H}_{n+2}(Y,X) \cong \pi_{n+2}(Y,X) \cong A$$

Therefore Y/X is $(n+1)$-connected and $\pi_{n+2}(Y/X) \cong A$.

Define $k: Y \longrightarrow Y/X \longrightarrow (Y/X)[n+2] = K(A, n+2)$

Then conclude by comparing the fiber sequence

$K(A,n+1) \to X \longrightarrow Y$ and $\mathrm{hoFib}(k) \to Y \longrightarrow K(A,n+2)$

to show $\mathrm{hoFib}(k) \simeq X$. "$\square$".

In the Postnikov-tower of a 1-connected space, all fiber sequence $K(\pi_{n+1}X, n+1) \to X[n+1] \xrightarrow{p_{n+1}} X[n]$ "deloop", i.e. are classified by a k-invariant

$k_n : X[n] \longrightarrow K(\pi_{n+1}X, n+2)$ which we see as

a class $k_n \in H^{n+2}(X[n]; \pi_{n+1}X)$

Given $\pi_2 X, \pi_3 X, \pi_4 X, \ldots, k_2, k_3, \ldots$ we can reconstruct inductively X. Start $X[2] = K(\pi_2 X, 2)$

$X[3] = \mathrm{hoFib}\,(X[2] \xrightarrow{k_2} K(\pi_3 X, 4))$, etc...

⚠ All p_n's are fibrations, and so $\mathrm{holim}(\cdots X[n+1] \twoheadrightarrow X[n] \twoheadrightarrow \cdots)$

$= \lim(\cdots X[n+1] \to X[n] \to \cdots) \overset{\text{w.e.}}{\simeq} X$ (not homotopy eq!)

§7. More cool stuff

(a) Poincaré sphere.

A_5 = group of orientation preserving isometries of a regular icosahedron

$$\pi_1 SO(3) \cong \mathbb{Z}/2$$

$$\begin{tikzcd}
A_5 \arrow[r, phantom, "<"] & SO(3) \\
120 \text{ elements } p^{-1}(A_5) = I \arrow[u, two heads] \arrow[r, phantom, "<"] & S^3 = \text{unit quaternions} \arrow[u, "p"'] \\
\mathbb{Z}/2 \arrow[u, hook]
\end{tikzcd}$$

- $H_1(K(I,1);\mathbb{Z}) = I_{ab} = 0$
  I perfect
  $H_2(K(I,1);\mathbb{Z}) = 0$
  I superperfect.

The free action of I on S^3 yields a covering space $S^3 \longrightarrow S^3/I = X$ path-connected space

- $H_1(X;\mathbb{Z}) = 0$ and $\pi_1 X \cong I$
- $H_2(X;\mathbb{Z}) = 0$ (superperfect)
- $H_3(X;\mathbb{Z}) \cong H_3(S^3;\mathbb{Z}) \cong \mathbb{Z}$.

This is Poincaré's <u>homology sphere</u>.

(b) James construction

There's a combinatorial model of $\Omega\Sigma X$, namely the "free topological monoid" $J(X)$. X is a based space

$X = J_1(X)$,

elements are words of length 1

$$\begin{tikzcd}
X \vee X \arrow[r, hook] \arrow[d, "\nabla"'] & X \times X \arrow[d] & & \\
J_1(X) = X \arrow[r, hook] & J_2(X) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, hook] & J_3(X) \arrow[r, hook] & \cdots
\end{tikzcd}$$

words of length 2

$$J(X) \simeq \operatorname{hocolim}\left(J_1(X) \hookrightarrow J_2(X) \hookrightarrow \cdots\right) \simeq \Omega\Sigma X.$$

Here $J_2(X)/X \simeq X \times X / X \vee X \simeq X \wedge X$

If X is $(n-1)$-connected, this is $(2n-1)$-connected.

In fact $X \hookrightarrow J(X)$ is $(2n-1)$-connected

On π_*: $\quad \pi_k X \longrightarrow \pi_k J(X) = \pi_k \Omega\Sigma X = \pi_{k+1}\Sigma X$

it induces the suspension map.

This map is iso for $k < 2n-1$. (Freudenthal)

ⓒ Infinite symmetric product (Dold–Thom)

free top. abelian monoid

$$X = SP^1(X) \longrightarrow X\times X/\Sigma_2 \longrightarrow X\times X\times X/\Sigma_3 \longrightarrow \cdots \quad SP^\infty(X)$$

The map $X \longrightarrow SP^\infty(X)$ induces

$$Hu: \pi_k X \longrightarrow \pi_k SP^\infty(X) \cong H_k(X;\mathbb{Z}).$$

Moreover $SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X;\mathbb{Z}); n)$

It's a generalized Eilenberg-MacLane space (GEM), all k-invariants are zero.

ⓓ Blakers-Massey

Consider a ho. p.o.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & B \arrow[dd] \\
 & P \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \\
C \arrow[rr] & & D
\end{tikzcd}$$

$P = \mathrm{hopb}(C \longrightarrow D \longleftarrow B)$

The comparison map $h: A \longrightarrow P$ is $(n+m-1)$-connected if the pair (B,A) is n-connected, (C,A) is m-connected.

$B = C = *$, $D = \Sigma A$, $P = \Omega\Sigma A$