Remark 2.5

① In fact $Hu: \pi_{n+1} X \longrightarrow H_{n+1}(X;\mathbb{Z})$ is surjective

② When $n=1$, X path-connected, $Hu: \pi_1 X \longrightarrow H_1(X;\mathbb{Z})$ is abelianization, $\operatorname{Ker} Hu = [\pi_1 X, \pi_1 X]$.

③ There is a relative version for $(n-1)$-connected pairs of 1-connected spaces (X,A). Here $Hu: \pi_n(X;A) \xrightarrow{\cong} H_n(X,A;\mathbb{Z})$

④ For $n \geq 2$, one can construct Moore spaces $M(A,n)$ with a single non-trivial reduced homology group, namely $H_n(M(A,n);\mathbb{Z}) \cong A$. This can be realized as an $(n-1)$-connected CW-cx of dimension $(n+1)$.
The n-cells correspond to generators of A,
$(n+1)$ —————— relations in A.

Then we can construct $K(A,n) = M(A,n)[n]$: indeed this is an $(n-1)$-connected space, $\pi_n K(A,n) \cong \pi_n M(A,n)$
$\overset{Hu}{\cong} H_n(M(A,n);\mathbb{Z}) \cong A$. Ex: $M(\mathbb{Z}/p, n) = S^n \cup e^{n+1}$

§ 3 Eilenberg-MacLane spaces and cohomology

Our objective is to compare in Top_*, $[-;K(A,n)]$ and $\tilde{H}^n(-;A)$. To compute the latter functor, take first your favorite chain complex for X (singular, cellular), say $\tilde{C}_*^{cell}(X)$ so that $\tilde{C}_n^{cell}(X) = \bigoplus_{n\text{-cells}} \mathbb{Z}$, $n>0$. Next construct a cochain complex by taking $Hom(\tilde{C}_*^{cell}(X); A) = \tilde{C}^*_{cell}(X;A)$. Then $\tilde{H}^n(X;A)$ is the n-th reduced cohomology group of that cochain complex.

Theorem 3.1

$[-;K(A,n)]$ is a contravariant, homotopy invariant, functor from Top_* to Ab which sends cofiber sequences $X \hookrightarrow Y \to Y/X$ to exact sequences. Moreover we have natural isos $[X;K(A,n)] \cong [\Sigma X; K(A,n+1)]$

proof: $[X, K(A,n)]$ is an abelian group because $K(A,n) \simeq \Omega^2 K(A,n+2)$. The natural isomorphisms come from $[\Sigma X, K(A,n+1)] \cong [X, \Omega K(A,n+1)]$

$$\underset{\Sigma - \Omega}{\overset{\text{adj}}{\cong}} [X, K(A,n)] \qquad \square$$

Together with the next lemma, this theorem says that the sequence $[-, K(A,n)]$, $n \geq 0$, satisfy the Eilenberg-Steenrod axioms for a reduced cohomology theory.

<u>Lemma 3.2</u>

$[\_, K(A,n)]$ satisfies the <u>wedge axiom</u>, i.e.

$$[\bigvee_{i\in I} X_i, K(A,n)] \cong \prod_{i \in I} [X_i, K(A,n)]$$

proof: $[\bigvee X_i, K(A,n)] \cong \pi_0 \operatorname{map}_*(\bigvee X_i, K(A,n)) \cong$

$$\underset{\substack{\bigvee : \text{ coproduct} \\ \text{in } Top_*}}{\cong} \pi_0\left(\prod \operatorname{map}_*(X_i, K(A,n))\right) = \prod \pi_0 = \prod_{i\in I} [X_i, K(A,n)]$$

$\square$

Lemma 3.3

$$[S^k; K(A,n)] \cong \begin{Bmatrix} A & \text{if } k=n \\ 0 & \text{if } k\neq n \end{Bmatrix} \cong \tilde{H}^n(S^k; A)$$

proof: The first iso is by definition of $K(A,n)$. The second one is by computation:

$\tilde{C}_*^{cell}(S^k)$: $\quad 0 \to \underset{k}{\mathbb{Z}} \to 0 \to \cdots \to \underset{0}{0}$

$\tilde{C}^*_{cell}(S^k; A)$ $\quad 0 \leftarrow \underset{\underset{A}{\cong}}{Hom(\mathbb{Z}; A)} \leftarrow 0 \cdots$

$\square$

Lemma 3.4

$$H^n(K(A,n); A) \cong Hom(A; A)$$

proof: $\tilde{C}_*^{cell}(K(A,n))$: $\cdots \to \underset{relation}{\overset{n+2}{}\ \overset{n+1}{\oplus \mathbb{Z}}} \xrightarrow{\partial} \underset{gen}{\overset{n}{\oplus \mathbb{Z}}} \to 0 \cdots \to \overset{0}{0}$

and $\operatorname{Coker}\partial \cong A$.

Apply $Hom(\_\,; A)$ $\qquad Hom(\underset{rel}{\oplus \mathbb{Z}}; A) \overset{\partial^*}{\leftarrow} Hom(\underset{gen}{\overset{n}{\oplus \mathbb{Z}}}; A) \leftarrow 0$

$$H^n(K(A,n); A) = \operatorname{Ker}\partial^* \underset{\text{left exact}}{\cong} Hom(\operatorname{Coker}\partial, A) \cong Hom(A,A)$$

$\square$

Def 3.5 $T : [X, K(A,n)] \longrightarrow H^n(X;A)$

$$f \longmapsto f^*(\mathrm{id}_A).$$

f induces $f^* : H^n(K(A,n);A) \longrightarrow H^n(X;A)$

$$\begin{array}{c} \cong \\ \mathrm{Hom}(A,A) \ni \mathrm{id}_A \longmapsto T[f] \end{array}$$

Proposition 3.6
T is an isomorphism on all finite dimensional CW-cxs

proof. First T is an iso on all spheres.

$$T : [S^k, K(A,n)] \longrightarrow H^n(S^k; A)$$

Any chosen generator in A corresponds to a homomorphism $\mathbb{Z} \xrightarrow{g} A$ and can be realized as $\gamma : S^n \longrightarrow \underset{\text{gen}}{\vee} S^n \subset K(A,n)$

$$T(\gamma) = \gamma^*(\mathrm{id}_A) = g \in \mathrm{Hom}(\mathbb{Z}, A) \cong H^n(S^n; A)$$

The wedge axiom allows to extend this iso to wedges of spheres.

To conclude we use the long exact sequences (cofiber sequences/Puppe) associated to $\bigvee_I S^n \to X^n \to X^{n+1} \to \bigvee_I S^{n+1}$

$$\begin{array}{ccccccc}
0 = [\bigvee S^{n+1}; K(A,n)] & \to & [X^{n+1}, K(A,n)] & \to & [X^n, K(A,n)] & \to & [\bigvee S^n, K(A,n)] \\
\cong \downarrow T & & \cong \downarrow T & & \text{induct} \cong \downarrow T & & \cong \downarrow T \\
0 = H^n(\bigvee S^{n+1}; A) & \to & H^n(X^{n+1}; A) & \to & H^n(X^n; A) & \to & H^n(\bigvee S^n; A)
\end{array}$$

We conclude by the 5-Lemma (here X^n: n-skeleton). $\square$

§4. Derived functors of inverse limit

Let $(\mathbb{N}, \leq)$ be the category associated to the poset:

$$0 \to 1 \to 2 \to 3 \to \cdots$$

The skeletal filtration gives a functor $X^\bullet : \mathbb{N} \to \mathrm{Top}_*$

Let now $A_\bullet : \mathbb{N}^{op} \to Ab$ be a tower of abelian groups: $A_0 \xleftarrow{f_1} A_1 \xleftarrow{f_2} A_2 \xleftarrow{f_3} A_3 \leftarrow \cdots$

Def 4.1 The limit of $A_\bullet$ is the subgroup of $\prod_{n\geq 0} A_n$ of $(a_n)_{n\geq 0}$ st $f_n(a_n) = a_{n-1}$ for all $n \geq 1$.

Remark 4.2 Elements in $\lim A_\bullet$ are compatible sequences in the tower. Therefore a homomorphism $B \xrightarrow{\beta} \lim A_\bullet$ corresponds to a commutative diagram of homomorphisms

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ B \arrow[r, "\beta_n"] \arrow[d, "id"'] & A_n \arrow[d, "f_n"] \\ B \arrow[r, "\beta_{n-1}"] \arrow[d] & A_{n-1} \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

So $\lim$ is right-adjoint to $c : Ab \longrightarrow Tower$

$$B \longmapsto B \xleftarrow{id} B \xleftarrow{id} B \xleftarrow{id} \cdots$$

Def 4.3 The *shift map* $sh : \prod A_n \longrightarrow \prod A_n$

$$(a_n)_{n \geq 0} \longmapsto (f_n(a_n))_{n \geq 1}$$

Since f_n is a group homomorphism, so is sh.

Lemma 4.4

$$\operatorname{Ker}(id - sh) = \lim A_\bullet$$

proof: The kernel consists of sequences (a_n) s.t. $a_{n-1} = f_n(a_n)$ for all $n > 0$. $\square$

Def 4.5 $\lim^1 A_\bullet = \mathrm{Coker}(\mathrm{id} - \mathrm{sh})$

Proposition 4.6
Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short sequence of towers (ie $0 \to A_n \to B_n \to C_n \to 0$ is exact for all n).
Then there is a six term exact sequence

$$0 \to \lim A_\bullet \to \lim B_\bullet \to \lim C_\bullet \to \lim^1 A_\bullet \to \lim^1 B_\bullet \to \lim^1 C_\bullet \to 0$$

proof:

$$\begin{tikzcd}
0 \arrow[r] & \lim A_\bullet \arrow[r] \arrow[d, hook] & \lim B_\bullet \arrow[r] \arrow[d, hook] & \lim C_\bullet \arrow[d, hook] \arrow[dddll, bend left] & \\
0 \arrow[r] & \prod A_n \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{sh}"] & \prod B_n \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{sh}"] & \prod C_n \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{sh}"] & 0 \\
0 \arrow[r] & \prod A_n \arrow[r] \arrow[d, two heads] & \prod B_n \arrow[r] \arrow[d, two heads] & \prod C_n \arrow[r] \arrow[d, two heads] & 0 \\
 & \lim^1 A_\bullet \arrow[r] & \lim^1 B_\bullet \arrow[r] & \lim^1 C_\bullet \arrow[r] & 0
\end{tikzcd}$$

Conclude by the Snake Lemma. $\square$

Lemma 4.7

If $A_\bullet$ is a tower s.t. $\exists k$ with $f_n : A_n \to A_{n-1}$ surjective for all $n > k$, then $\lim^1 A_\bullet = 0$

proof: We prove that $\mathrm{Coker}(id - sh) = 0 \iff id - sh$ is surjective.

Let $(a_n)_{n\geq 0} \in \prod_{n\geq 0} A_n$. We are looking for $(b_n)_{n\geq 0}$

s.t. $b_n - f_{n+1}(b_{n+1}) = a_n \quad \forall n.$ $(*_n)$

Since f_{k+1} is surjective, choose b_{k+1} s.t. $-f_{k+1}(b_{k+1}) = a_k$

Then choose $b_k = 0$ so $(*_k)$ is satisfied and go downwards: $b_{k-1} = a_{k-1}$ so $(*_{k-1})$ is satisfied, then $b_{k-2} = a_{k-2} + f_{k-1}(b_{k-1})$, etc. ... $(*)_0$.

Finally move up: Since f_{k+2} is surjective, the equation

$(*_{k+1})$ $\quad b_{k+1} - f_{k+2}(b_{k+2}) = a_{k+1}$ has a solution

and conclude by induction. $\square$

__Mittag-Leffler condition__: For each k, $\exists j \geq k$ s.t.

$\mathrm{Im}(A_i \to A_k) = \mathrm{Im}(A_j \to A_k)$ for all $i \geq j$.

Then $\lim^1 A_\bullet = 0$.

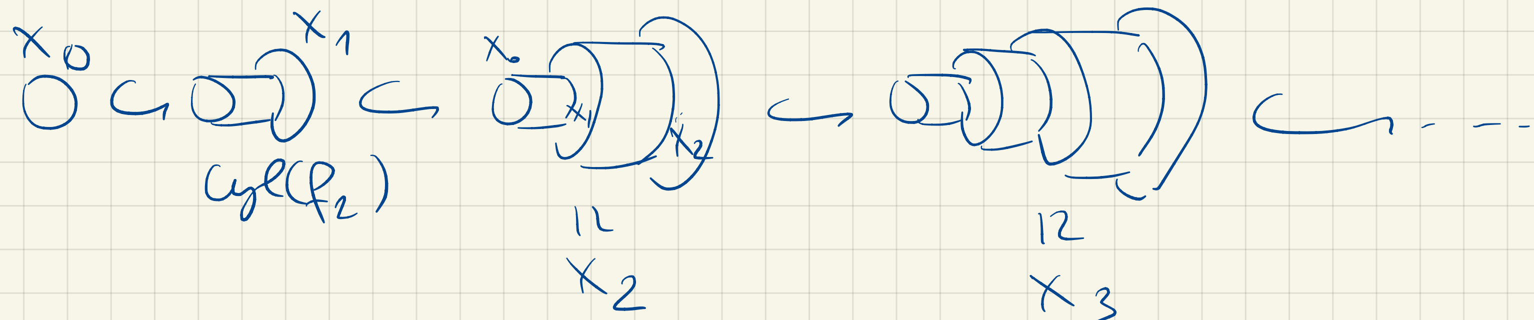
§ 5. Cohomology of CW-complexes

We wish to prove that $[X, K(A,n)] \cong H^n(X;A)$ for all CW-complexes X. We need to compute the value of such functors on $\bigcup_{n\geq 0} X^n$, where X^n is the n-skeleton. This is more general and let us assume that $X = \bigcup X_n$, where $X_n \subset X_{n+1}$ is a cofibration

Proposition 5.1
Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$ be a diagram $\mathbb{N} \longrightarrow Top_*$ for arbitrary maps f_n. The homotopy colimit of $X_\bullet$ can be computed as the strict colimit of a homotopy equivalent diagram where we turned every f_n into a cofibration

no proof

Remark 5.2 The standard model for the hocolim is the <u>telescope</u>. Change f_1 into $X_0 \hookrightarrow Cyl(f_1)$ then f_2 into $Cyl(f_1) \hookrightarrow Cyl(f_1) \cup_{X_1} Cyl(f_2)$, ...



We write $Tel(X_\bullet)$ for this construction.

By homotopy invariance of homotopy colimits, we have that the telescope of a diagram where all f_n's are <u>already</u> cofibrations coincides with $\bigcup X_n$.

$$\begin{tikzcd}
X_0 \arrow[r, hook, "f_1"] \arrow[d, "id"'] & X_1 \arrow[r, hook, "f_2"] & X_2 \arrow[r, hook] & X_3 \cdots & \bigcup X_n \\
X_0 \arrow[r, hook] & Cyl(f_1) \arrow[r, hook] \arrow[u, "\wr"'] & Cyl(f_1) \cup_{X_1} Cyl(f_2) \arrow[r, hook] \arrow[u, "\wr"'] & \cdots & Tel(X_\bullet) \arrow[u, "\wr"']
\end{tikzcd}$$