Remark 7.15 We have learned that a fibration yields a les in $\pi_*$, we have also seen that the mapping fiber construction gives us such a dual Puppe sequence. These are the same when starting with a fibration. Consider

$$\begin{tikzcd}
E \arrow[r, two heads, "p"] \arrow[d, "id_E", "\wr"'] & B \arrow[d, "id_B", "\wr"'] & b_0 \arrow[l] \arrow[d, "\wr", "c_{b_0}"'] & \overset{\text{pullback}}{\rightsquigarrow} & F_{b_0} = p^{-1}(b_0) \text{ fiber} \arrow[d] \\
E \arrow[r, two heads, "p"] & B & map_*(I, B) \arrow[l, two heads, "ev_1"'] & \overset{\text{pullback}}{\rightsquigarrow} & F(p) \text{ mapping fiber}
\end{tikzcd}$$

homotopy fiber

By homotopy invariance of homotopy pullback $F_{b_0} \xrightarrow{\simeq} F(p)$ is a homotopy eq. So we get the same les's !

# Chapter V, The Hurewicz Theorem

Here we study the relationship between $\pi_*$ and $H_*(-;\mathbb{Z})$

## § 1. Homotopical background

We will work with CW-complexes and (weak) homotopy equivalences because $\pi_n(-)$ are weak homotopy invariant, and $H_n(-;\mathbb{Z})$ as well.

**Proposition 1.1** (WRiteRead)
A weak homotopy equivalence $f: X \longrightarrow Y$ induces an iso on $H_n(X;\mathbb{Z}) \xrightarrow[\cong]{f_*} H_n(Y;\mathbb{Z})$, for all $n \geq 0$.

proof. [Hatcher's, Chapter 2]. We know that $H_n(-;\mathbb{Z})$ is a homotopy invariant functor. So we will study $f': X \hookrightarrow Y'$ where $Y' \simeq Y$, say $Y' = Cyl(f)$. To prove $H_n(X;\mathbb{Z}) \cong H_n(Y';\mathbb{Z})$, we prove $H_n(Y', X;\mathbb{Z}) = 0$ for all $n$ and conclude from the les in $H_*$ of a pair.

Let $\alpha$ be a homology class represented by a relative cycle $a = \sum k_i \sigma_i$, $\sigma_i : \Delta^n_i \longrightarrow Y'$, st $\partial a$ is a chain in $X$.

Construct a simplicial complex $K$ from all $\Delta^n_i$ by identifying pairs of faces on which the corresponding $\sigma_i$'s coincide.

The $\sigma_i$'s induce a map $\sigma : K \longrightarrow Y'$. Consider the subcomplex $L \subset K$ consisting of the faces appearing in $\partial a$.

So we can see $\sigma : (K, L) \longrightarrow (Y', X)$.

Moreover there is a cycle $b$ in $C_n^{sing}(K, L)$ whose image under $\sigma_*$ is $a$ (by taking $\Delta^n_i \xrightarrow{id} \Delta^n_i \to K$).

We assume $f$ is a w.e., so $f'$ is so, ie $\pi_n(Y', X) = 0$ for all $n$. Since $K$ is made from $L$ by attaching $n$-dim cells, we can use the compression lemma and find a map $\tau \simeq \sigma$ s.t. $\tau$ factors through $(X, X)$.

Finally we see $\alpha = \sigma_*[b] = \tau_*[b] = 0$. as it factors through $H_n(X, X; \mathbb{Z}) = 0$.

$\square$

**Lemma 1.2**

Let $X$ be a CW-complex and $X^{n+k}$ is its $(n+k)$-skeleton, $k \geq 1$.
Then $i : X^{n+k} \subset X$ induces an iso

- $\pi_n X^{n+k} \xrightarrow{\cong} \pi_n X$
- $H_n(X^{n+k}; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z})$

proof: • The pair $(X, X^{n+k})$ is $(n+k)$-connected, we get iso $\pi_i X^{n+k} \longrightarrow \pi_i X$ for $i \leq n+k-1$.

• We use cellular homology and see that in degrees $\leq n+1$ $C_*^{cell}(X^{n+k})$ and $C_*^{cell}(X)$ agree. $\square$

**Example 1.3** Let $n \geq 2$. As a CW-complex $X = S^n \times S^n$ has 4 cells $e^0 \times e^0$, $e^n \times e^0$, $e^0 \times e^n$, $e^n \times e^n \approx e^{2n}$. So $X^0 = *$, $X^n = S^n \vee S^n$, $X^{2n} = S^n \times S^n = X$.

In particular $\pi_n(S^n \vee S^n) \cong \pi_n(S^n \times S^n) \cong \mathbb{Z} \times \mathbb{Z}$

In fact generators of this free abelian group are given by wedge inclusions $j_1, j_2 : S^n \hookrightarrow S^n \vee S^n$, because they are so when composing all the way to $S^n \times S^n$.

More generally $\pi_n(\bigvee_\alpha S^n_\alpha) \cong \bigoplus_\alpha \mathbb{Z}_\alpha$, because the product is constructed from the wedge by attaching cells of dim $\geq 2n$. Remember that any map from a (compact) sphere into a product factors through a finite product.

<u>Remark 1.4</u> Let $X$ be an $(n-1)$-connected space for $n \geq 2$ (at least path-connected & 1-connected).

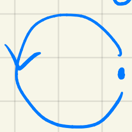
Then CW-approximation $X' \longrightarrow X$ allows us to replace $X$ by a CW-complex with $(X')^0 = *$ and $(X')^{n-1} = *$, $(X')^n = \bigvee_{i \in I} S^n_i$, where $I$ is a set of generators of $\pi_n X$

$(X')^{n+1}$ is obtained from $\bigvee_{i \in I} S^n_i$ by attaching $(n+1)$-cells: There is a map $f: \bigvee_{j \in J} S^n_j \longrightarrow \bigvee_{i \in I} S^n_i$ and $(X')^{n+1}$ is the pushout of

$$\bigvee_{j \in J} D^{n+1}_j \hookleftarrow \bigvee_{j \in J} S^n_j \xrightarrow{f} \bigvee_{i \in I} S^n_i .$$

The results we have seen here tell us that if we are interested in $\pi_n X$ and $H_n(X;\mathbb{Z})$, we might as well replace $X$ by $(X')^{n+1}$.

## §2. The Hurewicz homomorphism

We have $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Fix a generator $u_1 \in H_1(S^1;\mathbb{Z})$ by $u_1 =$ class of the cycle $\Delta^1 \longrightarrow S^1$, degree 1 map, anticlockwise:  . Excision induces isomorphisms

$$\begin{array}{ccccccc} H_1(S^1;\mathbb{Z}) & \cong & H_2(S^2;\mathbb{Z}) & \cong & H_3(S^3;\mathbb{Z}) & \longrightarrow \cdots & H_n(S^n;\mathbb{Z}) \\ u_1 & \longmapsto & u_2 & \longmapsto & u_3 & \longmapsto \cdots & u_n \end{array}$$

**Def 2.1** Let $X$ be a path-connected space. The Hurewicz homomorphism

$$\begin{aligned} Hu_X = Hu : \pi_n X &\longrightarrow H_n(X;\mathbb{Z}) \\ \alpha &\longmapsto Hu(\alpha) \end{aligned}$$

where $Hu(\alpha) = a_*(u_n)$ if $\alpha$ is represented by the map $a : S^n \longrightarrow X$, and $a_* = H_n(a;\mathbb{Z})$.

This is well defined since homotopic maps induce the same map in homology. It's a homomorphism: if $a, b : S^n \longrightarrow X$, the $\alpha + \beta = [a] + [b]$ is represented by $S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{a \vee b} X \vee X \xrightarrow{\nabla} X$

apply $H_n$

$$
\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(S^n \vee S^n) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(X \vee X) \arrow[r, "\nabla"] \arrow[d, phantom, "\cong"] & H_n X \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\Delta"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\alpha_* \oplus \beta_*"] & H_n X \oplus H_n X \arrow[r, "\nabla"] & H_n X \\
u_n \arrow[r, mapsto] & (u_n, u_n) \arrow[r, mapsto] & (Hu(\alpha), Hu(\beta)) \arrow[d, "\nabla"] & \\
 & & Hu(\alpha) + Hu(\beta) &
\end{tikzcd}
$$

**Proposition 2.2**

The Hurewicz homomorphism is natural in $X$

proof. Let $f : X \longrightarrow Y$. Consider

$$
\begin{tikzcd}
\pi_n X \arrow[r, "Hu_X"] \arrow[d, "f_*"'] & H_n(X; \mathbb{Z}) \arrow[d, "f_*"] \\
\pi_n Y \arrow[r, "Hu_Y"] & H_n(Y; \mathbb{Z})
\end{tikzcd}
$$

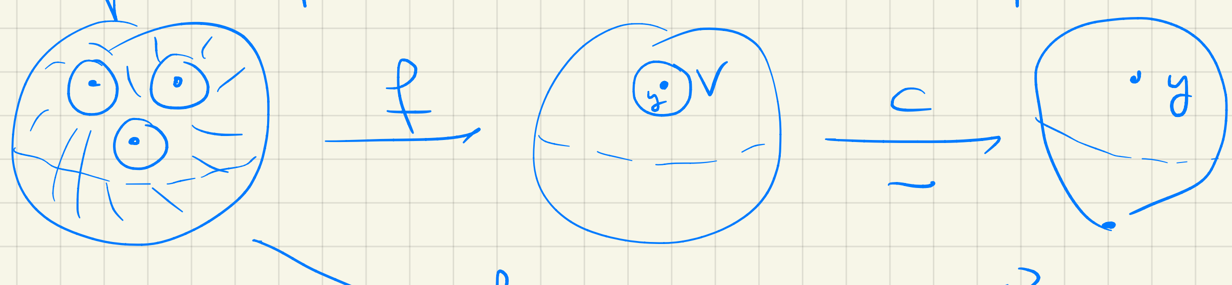
This square commutes since for $a : S^n \longrightarrow X$ the following square does so by functoriality of homology

$$
\begin{tikzcd}
u_n \arrow[r, mapsto] & Hu(\alpha) \arrow[dd, bend left, mapsto] & \\
H_n(S^n; \mathbb{Z}) \arrow[r, "\alpha_*"] \arrow[d, equal] & H_n(X; \mathbb{Z}) \arrow[d, "f_*"] & \\
 & & f_*(Hu(\alpha)) \arrow[d, phantom, "//"] \\
H_n(S^n; \mathbb{Z}) \arrow[r, "(f \circ \alpha)_*"] & H_n(Y; \mathbb{Z}) & \\
u_n \arrow[r, mapsto] & Hu(f_*(\alpha)) &
\end{tikzcd}
$$

$\square$

**Lemma 2.3** Let $n \geq 1$. The Hurewicz homomorphism $Hu: \pi_n S^n \to H_n(S^n; \mathbb{Z})$ is an iso

<u>proof</u>: Let us briefly recall how $\pi_n S^n \cong \mathbb{Z}$. This is based on the technical PL lemma: any map $S^n \to S^n$ can be deformed up to homotopy into one, $f$, which is PL on a polyhedron. If $f$ restricted to some simplex is not a bijection, then $f$ misses a point in the target, so factors through $S^n - *$ which is contractible. Then $f \simeq *$, so $Hu[f] = 0$.
If not, restriction to each simplex is given by an invertible matrix and we proceed like Hopf in the 1930's. Choose a point $y$ in the target and a small neighborhood $V \approx \mathring{D}^n$, the preimages $U_1, \dots, U_k$ are disjoint open balls in certain simplices of the polyhedron




Compose with the collapse map to $S^n / S^n \setminus V$. This factors through $\bigvee_1^k S^n$

So $f$ is homotopic to $S^n \xrightarrow{\text{pinch}} \bigvee_1^k S^n \xrightarrow{\vee \pm 1} \bigvee_1^k S^n \xrightarrow{\nabla} S^n$

Any map is homotopic to a sum of maps of degree 1 or $-1$. If $d$ is the sum of these $\pm 1$, then $f$ has degree $d$. The isomorphism

$$\begin{aligned} \pi_n S^n &\longrightarrow \mathbb{Z} \\ f &\longmapsto d \end{aligned}$$

What about $Hu$? $\pi_n S^n \longrightarrow H_n(S^n; \mathbb{Z})$. Since $\deg(id) = 1$,

$[id_{S^n}] \longmapsto u_n$ this is an iso

$\square$

**Theorem 2.4** (Hurewicz Theorem)
Let $n \geq 2$, $X$ be $(n-1)$-connected. Then $Hu: \pi_n X \to H_n(X; \mathbb{Z})$ is an iso.

proof. We can assume that $X$ is a CW-complex s.t. $X^n = \bigvee_i S^n$, $X = X^{n+1}$ ( $Hu_X$ is an iso iff it's so for $CW(X) \to X$ and then even $CW(X)^{n+1}$ ). In Example 1.3 we computed $\pi_n(\bigvee_i S^n_i) \cong \bigoplus_i \mathbb{Z}_i$ with generators $S^n \hookrightarrow \bigvee_{i \in I} S^n_i$, inclusion in wedge summand.

Let us look at $Hu[j_i]$:

$$S^n \xhookrightarrow{j_i} \bigvee_i S^n \quad \rightsquigarrow \quad H_n(S^n;\mathbb{Z}) \longrightarrow H_n(\bigvee_i S^n;\mathbb{Z}) \cong \bigoplus_i \mathbb{Z}_i$$

$$u_n \longmapsto (j_i)_*(u_n) = e_i = (0,\dots,0,1,0,\dots)$$

$Hu$ sends generators to generators. It's an iso.

In general we deal with $X = \mathrm{Cof}(\bigvee_j S^n \xrightarrow{f} \bigvee_i S^n)$

We have a commutative diagram, $i : X^n \subset X$

$$\begin{tikzcd}
\pi_n(\bigvee_j S^n) \arrow[r, "f_*"] \arrow[d, "Hu"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \pi_n(\bigvee_i S^n) \arrow[r, two heads, "i_*"] \arrow[d, "\cong", "Hu"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_n X \arrow[d, "Hu"] & \\
H_n(\bigvee_j S^n) \arrow[r, "f_*"] \arrow[d, equal] & H_n(\bigvee_i S^n) \arrow[r, two heads, "i_*"] \arrow[d, equal] & H_n(X) \arrow[r] \arrow[d, "\cong" sloped] & 0 \quad \text{exact} \\
C^{cell}_{n+1}(X) \arrow[r, "\partial"] \arrow[d, equal] & C^{cell}_n(X) \arrow[r] \arrow[d, equal] & \mathrm{Coker}\,\partial & \\
\bigoplus_j \mathbb{Z} & \bigoplus_i \mathbb{Z} & &
\end{tikzcd}$$

$$\alpha \longmapsto w' \longmapsto w, \qquad w \longmapsto 0$$

$$\alpha \longmapsto y, \qquad y \longmapsto f_*^{cell}(y) = x \longmapsto 0$$

We conclude by a diagram chase for $w \in \pi_n X$, s.t. $Hu(w) = 0$.

Note $i_*$ epi by cellular approximation. So $w = i_* f_*(\alpha) = 0$.

Since $i_*$ is epi, so is $Hu_X$.

$\square$