§7 Fibrations

We dualize what we've done for cofibrations via HEP and introduce "nice surjections" that satisfy a homotopy lifting property (HLP).

Def 7.1 A map $p: E \longrightarrow B$ has the HLP for a space X if for any map $f: X \longrightarrow E$, the total space, and any homotopy $H: X \times I \longrightarrow B$, the base space, there exists a lift in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

We suppose $p \circ f = H \circ i_0$ and F verifies $p \circ F = H$ and $F \circ i_0 = f$.

Def 7.2 A map p is a (Hurewicz) fibration if it has the HLP for all spaces X.

Note: A Serre fibration has the HLP for all cubes I^n, $\forall n \geq 0$.

Remark 7.3 Homotopy equivalences, cofibrations, and fibrations equip Top with a "Quillen model structure", (Quillen: 1967; Strøm: 1968). There's another model structure with weak equivalences, retracts of relative cell complexes, Serre fibrations.

Just like for cofibrations, there is a universal lifting problem, namely using a <u>path space</u> $P(p)$

$$\begin{tikzcd}
\mathrm{map}(I,E) \arrow[dr, dashed, "r"] \arrow[drr, bend left, "ev_0"] \arrow[ddr, bend right, "p_*"'] & & \\
& P(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& \mathrm{map}(I,B) \arrow[r, "ev_0"] & B
\end{tikzcd}$$

elements of $P(p)$ are pairs $(e, \beta) \in E \times \mathrm{map}(I,B)$ s.t. $\beta(0) = p(e)$

The universal property of the pullback gives us a map r.

Proposition 7.4
The map p is a fibration iff r has a section $s : P(p) \longrightarrow \mathrm{map}(I,E)$ s.t. $r \circ s = \mathrm{id}$

<u>proof</u>: Let us first translate the HLP

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"'] \arrow[ur, dashed, "F"] & B \end{tikzcd} \quad \overset{\text{adjunction}}{\rightsquigarrow} \quad \begin{tikzcd} X \arrow[dr, dashed, "F"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "H"'] & & \\ & \mathrm{map}(I,E) \arrow[r, "ev_0"] \arrow[d, "p_*"] & E \arrow[d] \\ & \mathrm{map}(I,B) \arrow[r, "ev_0"'] & B \end{tikzcd}$$

If p is a fibration, then it must have the HLP for $P(p)=X$, choose $f: P(p) \longrightarrow E$, $H: P(p) \longrightarrow \mathrm{map}(I,B)$ given by the pullback construction, then F is the section we wanted.

Conversely if s exists, then we can choose, for any X, the map

$$F: X \longrightarrow P(p) \xrightarrow{s} \mathrm{map}(I,E).$$

$\square$

<u>Proposition 7.5</u>

(i) composition of fibrations is a fibration

(ii) pullback of a fibration is a fibration

<u>proof</u>: (i)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dd] & Z \arrow[d, "q"] \\ & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"'] \arrow[ur, dashed, "1"] \arrow[uur, dashed, "2"] & B \end{tikzcd}$$

(ii)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & E' \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"'] \arrow[ur, dashed, "2"] \arrow[urr, dashed, "1"] & B' \arrow[r] & B \end{tikzcd}$$

2 by universal property of pullback

$\square$

Lemma 7.6
Let $i: A \hookrightarrow B$ be a (closed) cofibration of locally compact Hausdorff spaces. Then $i^*: map(B, Z) \to map(A, Z)$ is a fibration for any Z.

proof. We convert the HL problem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & map(B,Z) \arrow[d, "i^*"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "?"] & map(A,Z) \end{tikzcd}$$

into a new problem by adjunction

$$\begin{tikzcd} X \times A \arrow[r, "i_0"] \arrow[d, "X \times i"'] & X \times A \times I \arrow[d, "X \times i \times I"] \arrow[ddr, bend left, "H"] & \\ X \times B \arrow[r, "i_0"] \arrow[rrd, bend right, "f"'] & X \times B \times I \arrow[dr, dashed, "F"] & \\ & & Z \end{tikzcd}$$

Since i is a cofibration, so is $X \times i$, hence the map F exists. $\square$

Examples. ① $\{0\} \subset I$, $\{0, 1\} \subset I$ induce fibrations $map(I, Z) \xrightarrow{ev_0} Z$

$$map(I, Z) \xrightarrow{ev_{0,1}} Z \times Z.$$
$$\xi \longmapsto (\xi(0), \xi(1))$$

② $F \times B \xrightarrow{p_2} B$ is a fibration, called "trivial".

Proposition 7.8 (turning a map into a fibration)
Let $f : X \longrightarrow Y$ be any map. Then there exists a factorisation
$$f : X \xrightarrow{\simeq} P(f) \twoheadrightarrow Y$$
as a homotopy equivalence followed by a fibration.

proof: Consider the following pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}(I, Y) \arrow[d, two heads, "ev_{0,1}"] \\ X \times Y \arrow[r, "f \times Y"'] & Y \times Y \end{tikzcd}$$

The pullback of the fibration $ev_{0,1}$ is a fibration $P \twoheadrightarrow X \times Y$.

P consists of triples $(x, y, \beta) \in X \times Y \times \mathrm{map}(I, Y)$ st. $f(x) = \beta(0)$, $y = \beta(1)$. So in fact $P \cong P(f)$

$$(x, y, \beta) \longmapsto (x, \beta)$$
$$(x, \beta(1), \beta) \longleftarrow\!\shortmid (x, \beta)$$

Composing $P \twoheadrightarrow X \times Y \twoheadrightarrow Y$ gives the fibration we wanted. The homotopy equivalence is

$$\begin{array}{ccc} X & \longrightarrow & P(f) \\ x & \longmapsto & (x, \underbrace{c_{f(x)}}_{\text{constant path}}) \end{array}$$

$(\ldots)$ tom Dieck

$\square$

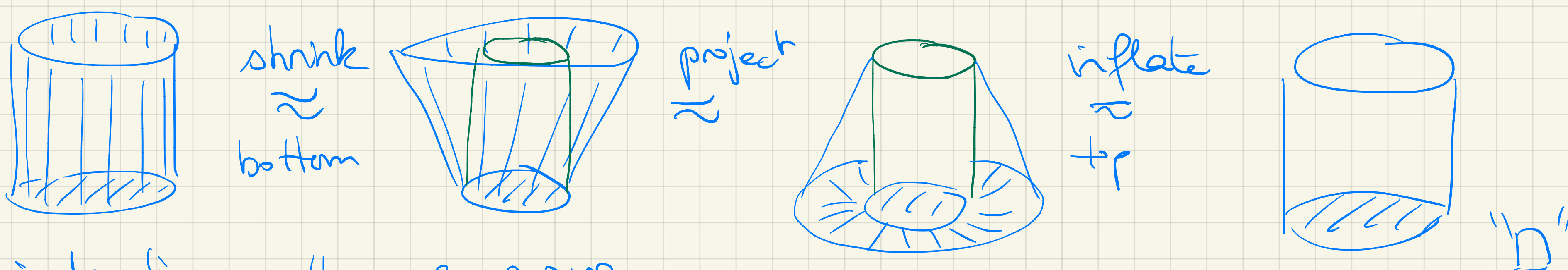
Sometimes we need to lift homotopies with fixed conditions on a subspace (cf. [Bredon]).

Lemma 7.10
Let p be a fibration. Any solid arrow commutative diagram

$$\begin{tikzcd} D^n \times 0 \cup S^{n-1} \times I \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a dotted lift.

proof: The pair $(D^n \times I, D^n \times 0 \cup S^{n-1} \times I)$ is homeomorphic to $(D^n \times I, D^n \times 0)$.

n=2: 

By induction cells we prove.

Corollary 7.11
Let $p: E \to B$ be a fibration, and $X \subset Y$ be a sub CW-complex, then the HL problem admits a solution.

$$\begin{tikzcd} Y \times 0 \cup X \times I \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Theorem 7.11

Let $i : X \subset Y$ be a strong deformation retract (of CW-cxs) and $p : E \to B$ be a fibration. Then any lifting problem of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\ Y \arrow[r, "g"'] \arrow[ur, dashed] & B \end{tikzcd}$$

has a dotted filler.

proof: The inclusion $i : X \hookrightarrow Y$ admits a retraction $r : Y \to X$ s.t. $r \circ i = id_X$ and $i \circ r$ is homotopic to id_Y relative to X. Let F be such a homotopy starting id_Y.

$$\begin{tikzcd} X \times 0 \arrow[r, hook] \arrow[d] & X \times I \cup Y \times 1 \arrow[r, "p_1 \cup r"] \arrow[d, hook] & X \arrow[r, "f"] \arrow[d, "i"] & E \arrow[d, "p"] \\ Y \times 0 \arrow[r, hook, "i_0"] & Y \times I \arrow[r, "F"'] \arrow[urrr, dashed, "H"] & Y \arrow[r, "g"'] & B \end{tikzcd}$$

The second square commutes since F ends at $i \circ r$. By Corollary 7.10 we have a dotted filler $H : Y \times I \longrightarrow E$. Then we define $h = H \circ i_0$. Then $p \circ H \circ i_0 = g \circ F \circ i_0 = g \circ id_Y = g$.

$H(x,0) = f \circ p_1(x,0) = f(x)$ $\square$

Remark: In general one can replace i with a cofibration which is a homotopy equivalence.

Theorem 7.13

Let $p: E \longrightarrow B$ be a fibration, $B_0 \subset B$ a subspace containing b_0, the base point. If $E_0 = p^{-1}(B_0) \subset E$, then p induces an isomorphism $\pi_n(E, E_0) \xrightarrow{\cong} \pi_n(B, B_0)$ for any $n \geq 1$.

Proof. ① p_* surjective. Let $b: (D^n, S^{n-1}) \longrightarrow (B, B_0)$ be any map and consider

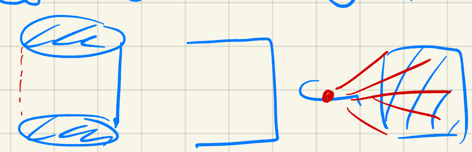
$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \arrow[r, "b"'] \arrow[ur, dashed, "e"] & B \end{tikzcd}$$

The lift e exists by Theorem 7.12 and represents a preimage of $[b]$ in $\pi_n(E, E_0)$.

② p_* is injective. Let $f, f': (D^n, S^{n-1}) \longrightarrow (E, E_0)$ be two maps s.t. $p \circ f \simeq p \circ f'$ via $H: D^n \times I \longrightarrow B$.

Consider

$$\begin{tikzcd} D^n \times \{0,1\} \cup * \times I \arrow[r, "(f,f') \cup e_0"] \arrow[d, hook] & E \arrow[d] \\ D^n \times I \arrow[r, "H"'] \arrow[ur, dashed] & B \end{tikzcd}$$



has a dotted by 7.12 again. So $f \simeq f'$. $\square$

Corollary 7.13

Let $F = F_{b_0} = p^{-1}(b_0)$ be the fiber over the base point. Then $\pi_n(E,F) \cong \pi_n B$. In particular we have a les in π_*

$$\pi_{n+1}F \xrightarrow{i_*} \pi_{n+1}E \xrightarrow{p_*} \pi_{n+1}B \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n E \to \cdots$$

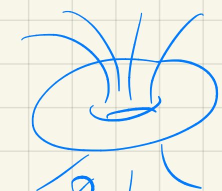
proof: The les in π_* for the pair (E,F) becomes the sequence indicated above. One can identify p_* as

$$\begin{tikzcd}
\pi_{n+1}(E,e_0) \arrow[r, "i_*"'] \arrow[d, phantom, "\cong" rotate=-90] & \pi_{n+1}(E,F) \arrow[r, "\cong", "p_*"'] & \pi_{n+1}(B,b_0) \arrow[d, phantom, "\cong" rotate=-90] \\
\pi_{n+1}E \arrow[rr] & & \pi_{n+1}B
\end{tikzcd}$$

$\square$

Example 7.14

The Hopf map $\eta: S^3 \to S^2$ can be seen as glued from two maps over hemispheres $D^2_+, D^2_- \subset S^2$, namely trivial fibrations $S^1 \times D^2 \to D^2$, where $S^3 = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$.



It's a fibration (...) with $\mathrm{Fib}(\eta) = S^1$.

The les gives $\pi_{n+1}S^1 \to \pi_{n+1}S^3 \xrightarrow{\cong} \pi_{n+1}S^2 \to \pi_n S^1 \to \pi_n S^3 \to \cdots$

But $\pi_n S^1 = 0$ $\forall n \geq 2$. Hence $\pi_{n+1}S^3 \cong \pi_{n+1}S^2$ $\forall n \geq 2$.

Of course $\pi_2 S^3 = 0$, and $\eta_*: \pi_3 S^3 \cong \mathbb{Z} \xrightarrow{\cong} \pi_3 S^2 \cong \mathbb{Z}$,
$1 \longmapsto [\eta]$