Back to injectivity

Pushout-product: Let $f : A \to B$, $g : X \to Y$ be two maps.

Construct

$$\begin{tikzcd} A \times X \arrow[r, "A\times g"] \arrow[d, "f\times X"'] & A \times Y \arrow[d, "f\times Y"] \arrow[dl] \\ B \times X \arrow[r, "B\times g"'] \arrow[r] & B \times Y \end{tikzcd}$$

$$A \times X \xrightarrow{A\times g} A\times Y \to P,\quad B\times X \to P \to B\times Y$$

The map $f \square g : P \to B \times Y$ is called the pushout-product of f and g.

General Fact: For maps in spaces, when f, g are cofibrations, so is $f \square g$. If f or g is a h.e, so $f \square g$.

Particular Case: $f : A \hookrightarrow B$ arbitrary cofibration, $g = \partial : \{0,1\} \subset I$

$$\begin{tikzcd} A \amalg A \arrow[r, hook, "(i_0, i_1)"] \arrow[d, "f \amalg f"'] & A \times I \arrow[d, "f\times I"] \arrow[dl] \\ B \amalg B \arrow[r, hook, "(i_0, i_1)"'] & B \times I \end{tikzcd}$$

$$B \amalg B \to P \to B \times I$$

$\boxed{\text{Claim: } P \hookrightarrow B \times I \text{ is a cofibration, which is a h.e if } f \text{ is so}}$

Note: $f \times I$ has the HEP for X since f has it for $\mathrm{map}(I, X)$

See: tom Dieck, Strøm, May's book

Note: A map $P \to X$ is a homotopy on A & two compatible maps from B

<u>Injectivity</u>; We assume $i : A \hookrightarrow B$ is a cofibration and
$\alpha : A \xhookrightarrow{\sim} A'$ is a cofibration and a homotopy equivalence. Let $h_0', h_1' : B' \longrightarrow X$, $B' = po(A' \xleftarrow{\sim} A \hookrightarrow B)$ s.t. $h_0' \circ \beta \simeq h_1' \circ \beta$ through $H : B \times I \longrightarrow X$.

Let $\alpha \,\square\, \partial : P \hookrightarrow A' \times I$ be the pushout-product map. Since α is a h.e, $\alpha \,\square\, \partial$ is a h.e. as well.

Let $\beta \,\square\, \partial : Q \hookrightarrow B' \times I$, it's also a cofibration.

The assumptions give us a map $H' : Q \longrightarrow X$.

Consider

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[d, hook, "\alpha \sqcup \alpha"'] & B \sqcup B \arrow[r, hook] \arrow[d, hook, "\beta \sqcup \beta"] & B \times I \arrow[d, hook] \arrow[ddr, bend left, "H"] & \\
A' \sqcup A' \arrow[r, hook] & B' \sqcup B' \arrow[r, hook] \arrow[ul, phantom, "\ulcorner"] \arrow[drr, bend right, "h_0', h_1'"'] & Q \arrow[ul, phantom, "\ulcorner"] \arrow[dr, bend left, "H'"] & \\
 & & & X
\end{tikzcd}$$

Composing pushouts gives a pushout (rectangle). Consider next

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[d, hook] & A \times I \arrow[r, hook] \arrow[d, hook] & B \times I \arrow[d, hook] \\
A' \sqcup A' \arrow[r, hook] & P \arrow[r, hook] \arrow[ul, phantom, "\ulcorner"] & Q
\end{tikzcd}$$

From the argument above (rectangle), we see that the right hand square is also a pushout square.

Finally

$$\begin{tikzcd}
A \times I \arrow[r, hook] \arrow[d, hook, "i \times I"'] \arrow[dr, phantom, "\ulcorner"] & P \arrow[d, hook] & A' \times I \arrow[l, hook', "\sim", "\alpha \square \partial"'] \arrow[d] \\
B \times I \arrow[r, hook] & Q & B' \times I \arrow[l, hook', "\beta \square \partial"]
\end{tikzcd}$$

For the same reason, the righthand square is a pushout

$H \circ (i \times I)$ is a homotopy on A and since $\alpha \times I$ is a h.e, we know $[A' \times I, X] \cong [A \times I, X]$. We get a map $K : A' \times I \to X$ s.t. $K \circ (\alpha \times I) \simeq H \circ (i \times I)$.

But α, and $\alpha \times I$, are cofibrations, so we can strictify and assume $K \circ (\alpha \times I) = H \circ (i \times I)$

$$\begin{tikzcd}
 & A \times I \arrow[d] & & \\
A' \amalg A' \arrow[r] & P \arrow[r, hook] \arrow[d, hook] & A' \times I \arrow[d, hook] \arrow[ddr, bend left, "K"] & \\
 & Q \arrow[r, hook] \arrow[drr, bend right, "H'"'] & B' \times I \arrow[dr, dashed, "\tilde{H}"] & \\
 & & & X
\end{tikzcd}$$

This diagram commutes because P being a pushout it's enough to check K and H' agree on $A \times I$ and $A' \amalg A'$. Hence we have an induced map $\tilde{H} : B' \times I \to X$.

It's precisely a homotopy $h_0' \simeq h_1'$.

- If α is arbitrary, factorize $\alpha : A \underset{j}{\overset{\sim}{\hookrightarrow}} Cyl(\alpha) \underset{q}{\xrightarrow{\sim}} A'$ ... $\square$

r retract & cof.

Lemma 5.9

Let us complete a diagram of solid arrows into a green cube

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dd] \arrow[dr] & & A_2 \arrow[dd] \arrow[dr] & \\
& A_3 \arrow[rr] \arrow[dd] & & A_{23} \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr] & & A_{12} \arrow[dr] & \\
& A_3' \arrow[rr] & & A_{23}'
\end{tikzcd}$$

by constructing push-outs A_{23}, A_{12}, A_{23}' (pushout of the bottom face).

Then the front face is also a pushout.

proof: Composing back and bottom gives a pushout rectangle

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & A_3' \arrow[d] \\
A_2 \arrow[r] & A_{23}'
\end{tikzcd}$$

. Since the top face is a pushout, so is the front face. $\square$

Lemma 5.10

Let $i: A \hookrightarrow B$ be a cofibration, $f: A \to C'$ any map.

Factor $f: A \overset{j}{\hookrightarrow} C \xrightarrow[\gamma]{\simeq} C'$. Then

$$\begin{tikzcd}
B \arrow[d, equal] & A \arrow[l, hook', "i"'] \arrow[r, hook, "j"] \arrow[d, equal] & C \arrow[d, "\wr\gamma"] \\
B & A \arrow[l, hook', "i"'] \arrow[r, "f"] & C'
\end{tikzcd}$$

gives homotopy equivalent pushout $D \simeq D'$.

proof.

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr] \arrow[dd, equal] & & B \arrow[dr] \arrow[dd, equal] & \\
& C \arrow[rr, hook] \arrow[dd, "\sim"'] & & D \arrow[dd, "?"] \\
A \arrow[rr, hook] \arrow[dr] & & B \arrow[dr] & \\
& C' \arrow[rr, hook] & & D'
\end{tikzcd}$$

Since the back face with two identities is a pushout, Lemma 5.9 applies and implies that the front face is a pushout. The pushout of the homotopy eq. γ along the cofibration $C \hookrightarrow D$ is a homotopy eq. by left properness. $\square$

Theorem 5.11 (Homotopy invariance of p.o.)
Consider a natural transformations of pushout diagrams

$$\begin{tikzcd}
C \arrow[d, "\gamma", "\wr"'] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\alpha", "\wr"'] & B \arrow[d, "\beta", "\wr"'] & \arrow[r, squiggly, "p.o."] & D \arrow[d, "\delta"] \\
C' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & B' & \arrow[r, squiggly, "p.o."] & D'
\end{tikzcd}$$

where i, i' are cofibrations, α, β, γ are homotopy equivalences. Then the induced map δ on horizontal pushouts is a homotopy equivalence.

Remark: It does not matter whether i' or f' is a cofibration.

proof. Factoring simultaneously f and f' into cofibrations followed by a homotopy equivalence, we can use Lemma 5.10 so as to reduce the proof to the case where f and f' are cofibrations. Then we factor the natural transformation as follows:

$$\begin{tikzcd}
C \arrow[d, "\gamma", "\wr"'] & A \arrow[l, hook', "f"'] \arrow[r, hook, "i"] \arrow[d, "\alpha", "\wr"'] & B \arrow[d, "\pi"', "\wr"] \arrow[dd, bend left, "\beta", "\wr"'] \\
C' \arrow[d, equal] & A' \arrow[l, hook', "f'"'] \arrow[r, hook] \arrow[d, equal] & P \arrow[d, "\wr"] \\
C' & A' \arrow[l, hook', "f'"'] \arrow[r, hook, "i'"] & B'
\end{tikzcd}
\quad \overset{\text{p.o}}{\rightsquigarrow} \quad
\begin{tikzcd}[row sep=small, column sep=small]
A \arrow[rr, hook] \arrow[dr, hook] \arrow[ddd, "\wr"'] & & B \arrow[dr, hook] & \\
& C \arrow[rr, hook, "\wr"] \arrow[ddd, "\wr"] & & D \arrow[ddd] \\
& & P \arrow[from=uu] & \\
A' \arrow[dr, hook] \arrow[rr, hook] & & & \\
& C' \arrow[rr, hook] & & D''
\end{tikzcd}$$

Here $P = \mathrm{po}(A' \xleftarrow{\alpha} A \xhookrightarrow{i} B)$, so $\pi : B \xrightarrow{\sim} P$ is a homotopy eq. by left properness. From the universal property of the p.o we get a map $P \to B'$ which is a homotopy eq. because β and π are so. From Lemma 5.10 we know that the two bottom pushouts are homotopy equivalent. We are left with the top part of the diagram; Lemma 5.9 implies $D \to D''$ is a h.e by left properness. □

Remark 5.12. We understood that the homotopy pushout of a diagram $C \xleftarrow{g} A \xrightarrow{f} B$ can be constructed by changing a single map into a cofibration (Lemma 5.10). In particular the double mapping cylinder has the same homotopy as $\text{p.o.}\left(C \xleftarrow{g} A \hookrightarrow Cyl(f) \right)$

§6. Playing with homotopy pushouts

Let's start with strict colimits.

Proposition 6.1 (Fubini for colimits)

Let I, J be small categories and $F : I \times J \longrightarrow Top$.

Then

$$\operatorname{colim}_{j \in J} \left[\operatorname{colim}_{I} F(\_, j) \right] \cong \operatorname{colim}_{i \in I} \left[\operatorname{colim}_{J} F(i, -) \right]$$

---

proof: Both are models for $\operatorname{colim}_{I \times J} F$.

Example 6.2 $X \vee Y / _{A \vee B} \simeq X/_A \vee Y/_B$

$A \subset X$, $B \subset Y$

$$\begin{tikzcd}
Y/B & \ast \arrow[l] \arrow[r] & X/A & \\
\ast \arrow[u, squiggly] & \ast \arrow[l, equal] \arrow[r, equal] \arrow[u, squiggly] & \ast \arrow[r, squiggly] \arrow[u, squiggly] & \ast \\
B \arrow[u] \arrow[d, hook] & \ast \arrow[l] \arrow[r] \arrow[u, equal] \arrow[d, equal] & A \arrow[u] \arrow[d, hook] \arrow[r, squiggly] & A \vee B \arrow[u] \arrow[d] \\
Y & \ast \arrow[l] \arrow[r] & X \arrow[r, squiggly] & X \vee Y
\end{tikzcd}$$

We wish to obtain an analogous statement for homotopy pushouts. We are looking at diagrams indexed by $I \times I$, where $I = \{2 \leftarrow 0 \rightarrow 1\}$, i.e.

$$\begin{tikzcd}
A_{12} & A_{10} \arrow[l] \arrow[r] & A_{11} \\
A_{02} \arrow[u] \arrow[d] & A_{00} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_{01} \arrow[u] \arrow[d] \\
A_{22} & A_{20} \arrow[l] \arrow[r] & A_{21}
\end{tikzcd}$$

Let us turn <u>all</u> maps into cofibrations and also do it so as to get cofibrations from the pushout of every square into the corner. For example consider the lower right square

1. Factor first α and β through their cylinders, $Cyl(\alpha) = A_{01}'$, $Cyl(\beta) = A_{20}'$
2. Construct the pushout P
3. Factor the map $P \longrightarrow A_{21}$ through its cylinder A_{21}'.

$$\begin{tikzcd}
A_{00} \arrow[rr, "\alpha"] \arrow[dr, hook] \arrow[d, hook] \arrow[dd, bend right, "\beta"'] & & A_{01} \arrow[dd] \\
Cyl(\beta) \arrow[d, "\sim"] & Cyl(\alpha) \arrow[ur, "\sim"'] & \\
A_{20} \arrow[rr] & & A_{21}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
A_{00} \arrow[r, hook] \arrow[d, hook] & A_{01}' \arrow[d, hook] \arrow[ddr, bend left] & \\
A_{20}' \arrow[r, hook] \arrow[drr, bend right] & P \arrow[dr] & \\
& & A_{21}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
A_{00} \arrow[r, hook] \arrow[d, hook] & A_{01}' \arrow[d, hook] \arrow[dd, bend left, hook] \\
A_{20}' \arrow[r, hook] \arrow[dr, hook] & P \arrow[d, hook] \\
& A_{21}'
\end{tikzcd}$$

Lemma 6.3

In this situation the horizontal pushouts induce a cofibration $A_0 \hookrightarrow A_1$, $A_0 \hookrightarrow A_2$.

proof.

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, hook] & & B \arrow[dr, hook] \arrow[dd, hook] \arrow[drr, bend left] & & \\
& C \arrow[rr, hook] \arrow[dd, hook] & & D \arrow[r] \arrow[dd, "\delta"] & P(X) \arrow[dd, "ev_0"] \\
A' \arrow[rr, hook] \arrow[dr, hook] & & B' \arrow[dr, hook] \arrow[urr, dashed, "\exists"] & & \\
& C' \arrow[rr, hook] & & D' \arrow[r] \arrow[uur, dashed, "?"] & X
\end{tikzcd}$$

Check the HEP for δ, by adjunction

P is a p.o. we get a map $P \longrightarrow P(X)$
since $P \hookrightarrow C'$ is a cof., get $C' \longrightarrow P(X)$
They induce a map on D'. $\square$

Proposition 6.4 (Fubini)

Let $A_{\bullet\bullet}$ a diagram on $I \times I$. Then

$$\operatorname{hopo}_{i \in I} \left(\operatorname{hopo}_I A_{\bullet i} \right) \simeq \operatorname{hopo}_{i \in I} \left(\operatorname{hopo} A_{i \bullet} \right)$$

proof: Horizontal pushout yields a po diagram of cofibrations $\square$

Example 6.4 Consider a pair of composable maps f and g. We claim that there is a homotopy cofiber sequence

$$C(f) \xrightarrow{\alpha} C(g\circ f) \longrightarrow C(g).$$

The map α is induced by naturality via

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, dashed] \arrow[d, "g"] & C(f) \arrow[d, "\alpha"] \\
A \arrow[r, "g\circ f"] & C \arrow[r, dashed] & C(g\circ f)
\end{tikzcd}$$

So we prove that $C(\alpha) \simeq C(g)$.

__proof:__ We consider the $I\times I$-diagram of which we take horizontal pushouts first and finally a vertical one.

$$\begin{tikzcd}
* \arrow[r, equal] \arrow[d, equal] & * \arrow[r, equal] & * \arrow[r, rightsquigarrow, "\text{hop.o.}"] & * \\
* & A \arrow[l] \arrow[r, "f"] \arrow[u] \arrow[d, equal] & B \arrow[r, rightsquigarrow] \arrow[u] \arrow[d, "g"] & C(f) \arrow[u] \arrow[d, "\alpha"] \\
* \arrow[u, equal] \arrow[d, rightsquigarrow, "\text{hopo}"] & A \arrow[l] \arrow[r, "g\circ f"] \arrow[d, rightsquigarrow] & C \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & C(g\circ f) \arrow[d, rightsquigarrow, "\text{hopo}"] \\
* \arrow[r, equal] & * \arrow[r] & C(g) \simeq & C(\alpha)
\end{tikzcd}$$

Alternatively we can take first vertical homotopy pushouts, of which we take horizontal hopo.

By Fubini we conclude that $C(g) \simeq C(\alpha)$. $\square$