Assume we have $h_0, h_1 : Z \longrightarrow Z'$ st. $f' \circ h_0 \simeq f \simeq f' \circ h_1$

Again we look at these maps as maps

$$\begin{tikzcd} Z \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] & & \\ Z' \arrow[r, hook, "i"] & Cyl(f') \arrow[r, "\simeq"] & X \end{tikzcd}$$

Since $i \circ h_0 \simeq i \circ h_1$, we have a homotopy H from $i \circ h_0 \simeq i \circ h_1$ and look at it as a map of pairs

$$H : (Z \times I,\ z_0 \times I \cup Z \times \{0,1\}) \longrightarrow (Cyl(f'), Z')$$

The Compression Lemma provides a homotopic homotopy $H' : Z \times I \longrightarrow Z'$. Hence $h_0 \simeq h_1$. □

Theorem 4.2

The CW-approximation is unique up to homotopy

proof: Let Z, Z' be two CW-approximations of X. We have $h : Z \longrightarrow Z'$ s.t. $f' \circ h \simeq f$. We also have $h' : Z' \longrightarrow Z$ st $f \circ h' \simeq f'$.

The composition $h' \circ h : Z \longrightarrow Z$ verifies

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "h' \circ h"'] \arrow[dr, phantom, "\simeq"] & X \\ Z \arrow[ur, "f"'] & \end{tikzcd}$$

commutes up to homotopy: $\underbrace{f \circ h'}_{\simeq f'} \circ h \simeq f' \circ h \simeq f$.

The identity also does the job. By uniqueness $h' \circ h \simeq id_Z$ □

We have used the following general fact about "equivalences"

Proposition 4.3

A map $\alpha : A \to A'$ is a homotopy equivalence iff it induces a bijection $\alpha^* : [A', X] \xrightarrow{\cong} [A, X]$ for any space X.

proof: $(\Rightarrow)$: α has a homotopy inverse α', then $\alpha \circ \alpha' \simeq id_{A'}$, $\alpha' \circ \alpha \simeq id_A$, inducing the identity on $[-, X]$.

$(\Leftarrow)$: We use $X = A$ and $id_A : A \to A$, there exists $\alpha' : A' \to A$ s.t. $\alpha' \circ \alpha \simeq id_A$. Now we look at

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & A' \\ A' \arrow[ur, "\alpha \circ \alpha'"'] & \end{tikzcd}$$

, which commutes up to homotopy since

$\alpha \circ \underbrace{\alpha' \circ \alpha}_{= id_A} \simeq \alpha$. Using $id_{A'}$ also solves the same problem. Since α^* is a bijection, $id_{A'} \simeq \alpha \circ \alpha'$. $\square$

Theorem 4.4 (Whitehead Theorem)

A weak homotopy equivalence between CW-complexes is a homotopy equivalence.

proof: f a w-e. $X \to X'$, $id_{X'}$ is a CW-approx. Apply 4.2 $\square$

§5. Homotopy pushouts

Let I be the category $2 \leftarrow 0 \rightarrow 1$. We have an adjunction

$$\mathrm{colim}_I : \mathrm{Top}^I \underset{\perp}{\rightleftarrows} \mathrm{Top} : c \quad \text{— constant diagram}$$

$$\mathrm{mor}_{\mathrm{Top}}(\mathrm{colim}_I F, X) \cong \mathrm{mor}_{\mathrm{Top}^I}(F, cX)$$

$$\delta \longleftrightarrow \text{“} h = (h_B, h_C) \text{”}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left, "h_B"] & \\
C \arrow[r] \arrow[drr, bend right, "h_C"'] & D \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\exists!\delta"] & \\
 & & X
\end{tikzcd}$$

Example 5.1 Homotopy invariance of pushouts

$$\begin{tikzcd}
F \arrow[d, "\alpha"'] & D^n \arrow[d, "\simeq"'] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "id"] & D^n \arrow[d, "\simeq"] & \arrow[r, squiggly, "p.o."] & & S^n \arrow[d, "\not\simeq"] \\
F' & * & S^{n-1} \arrow[l] \arrow[r] & * & & & *
\end{tikzcd}$$

We look at the vertical maps as a natural transformation between diagrams. This problem motivates us to adapt the construction of pushouts to get a homotopy meaningful version.

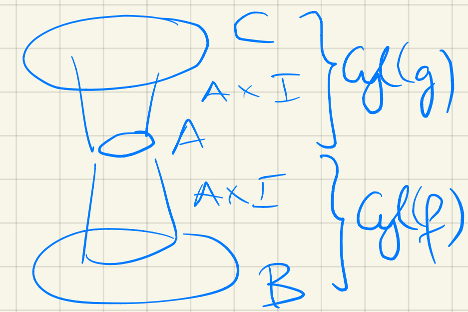
Def. 5.2 Let $C \xleftarrow{g} A \xrightarrow{f} B$ be a pushout diagram. Turn f and g into cofibrations, $f: A \overset{i}{\hookrightarrow} B' \xrightarrow{\sim} B$ and $g: A \overset{j}{\hookrightarrow} C' \xrightarrow{\sim} C$. The homotopy pushout of $C \xleftarrow{g} A \xrightarrow{f} B$ is the pushout of $C' \overset{j}{\hookleftarrow} A \overset{i}{\hookrightarrow} B'$.

Remark 5.3 We have a natural transformation (vertical)

$$\begin{tikzcd} C' \arrow[d, "\wr"'] & A \arrow[l, hook', "j"'] \arrow[r, hook, "i"] \arrow[d, "\mathrm{id}_A"] & B' \arrow[d, "\wr"] \\ C & A \arrow[l, "g"'] \arrow[r, "f"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \text{hopo} \arrow[d] \\ \text{po} \end{tikzcd}$$

We always have an induced map from the homotopy po to the usual po. We want to prove that hopo is a homotopy invariant and well-defined construction. In fact it's the best such construction for pushout diagrams mapping to the pushout: it's called the total left derived functor of the pushout.

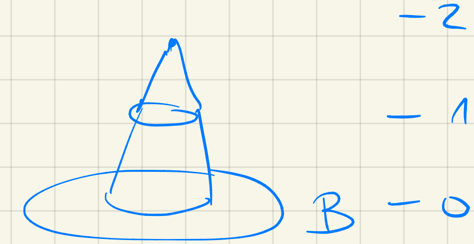
Example 5.4 The double mapping cylinder is the standard model of homotopy pushouts. We turn f, g into cofibrations using the mapping cylinder:



$$Cyl(g) \hookleftarrow A \hookrightarrow Cyl(f) \rightsquigarrow$$

For example if $C = *$, we get

$$CA \hookleftarrow A \hookrightarrow Cyl(f) \rightsquigarrow$$



We see, after reparametrization, that the homotopy pushout is homeomorphic to $C(f)$, also called homotopy cofiber (up to homotopy version of the cofiber $B/f(A)$)

For example, if $B = *$, we start with $* \leftarrow A \rightarrow *$, we modify it to $CA \hookleftarrow A \hookrightarrow CA$, assemble the parts to get ΣA. In the unpointed category we get the (unreduced) suspension, in the pointed category we get the reduced suspension using $A \rtimes I$ instead of $A \times I$.

Lemma 5.5 (Strictification)

Let $i : A \hookrightarrow B$ be a cofibration and a triangle
$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[dl, "g"] \\ X & \end{tikzcd}$$
which commutes up to homotopy. Then there exists $\bar{g} : B \longrightarrow X$ st. $\bar{g} \simeq g$ and $\bar{g} \circ i = f$.

proof: Let $F : A \times I \longrightarrow X$ be a homotopy from $g \circ i$ to f. Consider
$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "i_0"'] & B \arrow[d, hook, "i_0"] \arrow[ddr, bend left, "g"] & \\ A \times I \arrow[r, hook, "i \times I"] \arrow[drr, bend right, "H"'] & B \times I \arrow[dr, dashed, "G"] & \\ & & X \end{tikzcd}$$
, which commutes since $H(-,0) = g \circ i$. The HEP for i yields a homotopy $G : B \times I \longrightarrow X$ st. $G(i(a), 1) = H(a, 1) = f(a)$. We set $\bar{g} = G(-,1)$. $\square$

An important feature of topological spaces is left properness.

Proposition 5.6 (Left Properness)

Let $i : A \hookrightarrow B$ and $\alpha : A \longrightarrow A'$ a homotopy equivalence. The pushout of α along i is a map $\beta : B \longrightarrow B'$ which is a homotopy equivalence.

proof: To check that β is a homotopy equivalence, we prove $\beta^*: [B', X] \xrightarrow{\cong} [B, X]$ is a bijection for any space X.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "\alpha", "\wr"'] & B \arrow[d, "\beta", "?"'] \\ A' \arrow[r, hook, "i'"] & B' \end{tikzcd}$$

Surjection. Let $h: B \longrightarrow X$. Since α is a homotopy eq, there is a map $k: A' \longrightarrow X$ s.t. $h \circ i \simeq k \circ \alpha$.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "\alpha", "\wr"'] & B \arrow[d, "\beta"] \arrow[ddr, bend left, "h"] & \\ A' \arrow[r, hook, "i'"] \arrow[drr, bend right, "k"'] & B' \arrow[dr, dashed, "?"] & \\ & & X \end{tikzcd}$$

By the strictification lemma we change h to $\tilde{h} \simeq h$ s.t. $\tilde{h} \circ i = k \circ \alpha$

The universal property of the po yields a map $h': B' \longrightarrow X$ s.t. $h' \circ \beta = \tilde{h} \simeq h$.

Injectivity: Assume $h_0', h_1' : B' \longrightarrow X$ s.t. $h_0' \circ \beta \simeq h_1' \circ \beta$. Choose a homotopy $H: B \times I \longrightarrow X$ from $h_0' \circ \beta$ to $h_1' \circ \beta$. As $\_ \times I$ is a left adjoint, we have a pushout diagram of cylinders

$$\begin{tikzcd} A \times I \arrow[r, hook, "i \times I"] \arrow[d, "\alpha \times I"', "\wr"] & B \times I \arrow[d, "\beta \times I"] \arrow[ddr, bend left, "H"] & \\ A' \times I \arrow[r, hook, "i' \times I"] \arrow[drr, bend right, "K"'] & B' \times I \arrow[dr, dashed, "?"] & \\ & & X \end{tikzcd}$$

Because $\alpha \times I$ is a homotopy equivalence, there is $K: A' \times I \longrightarrow X$ st. $K \circ (\alpha \times I) \simeq H \circ (i \times I)$.

As before we change H for $\tilde{H}$ s.t. $\tilde{H} \simeq H$ and the diagram commutes strictly. We used $i \times I$ is a cofibration and notice that $\tilde{h}_0 = \tilde{H}(-,0) \simeq h_0$ by restriction of the homotopy for $\tilde{H}$ and H.

By universal property of the pushout we get $H' : B' \times I \longrightarrow X$ s.t. $H' \circ (i' \times I) = K$ and $H' \circ i_0 = \tilde{H}$.

What is $H'(-,0)$? (not equal to h_0') Consider

$$\begin{tikzcd}
A \arrow[r, "\alpha", "\sim"'] & A' \arrow[r, hook, "i'"] \arrow[d, "i_0"'] & B' \arrow[d, "i_0"] \arrow[ddr, bend left, "h_0'"] & \\
& A' \times I \arrow[r, hook, "i' \times I"] \arrow[drr, bend right, "K"'] & B' \times I \arrow[dr, "H'"] & ? \\
& & & X
\end{tikzcd}$$

<u>Question</u> $h_0' \circ i' \simeq K \circ i_0$?

Precompose with α, where we know they are equivalent.

$\square$